TGD as a Generalized Number Theory II: Quaternions, Octonions, and their Hyper Counterparts

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Abstract

This chapter is the second part of the multi-chapter devoted to the vision about TGD as a generalized number theory.

1. Hyper-quaternions and octonions

The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional number fields of quaternions and octonions.

The difficulties caused by the Euclidian metric signature of the number theoretical norm have however forced to give up the original idea as such, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The problem is that $H = M^4 \times CP_2$ cannot be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces are assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with fixed complex structure are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

2. Space-time-surface as a HQ or CHQ sub-manifold of hyper-octonionic imbedding space?

Space-time identified as a hyper-quaternionic (HQ) or co-hyper-quaternionic (coHQ) sub-manifold of the hyper-octonionic space in the sense that the tangent space or normal space of the space-time surface defines a hyper-quaternionic sub-algebra of the hyper-octonionic tangent space of H at each space-time point, looks an attractive idea. Second possibility is that the tangent space-algebra of the space-time surface is either associative or co-associative at each point.

One can also consider possibility that the dynamics of the space-time surface is determined from the requirement that space-time surface is algebraically closed in the sense that tangent space at each point has this property. Also the possibility that the property in question is associated with the normal space at each point of X^4 can be considered. Some delicacies are caused by the question whether the induced algebra at X^4 is just the hyper-octonionic product or whether the algebra product is projected to the space-time surface. If normal part of the product is projected out the space-time algebra closes automatically.

The first guess would be that space-time surfaces are hyper-quaternionic (HQ) or co-hyper-quaternionic (coHQ) sub-manifolds of hyper-octonionic space $HO = M^8$ with the property that complex structure is fixed and same at all points of space-time surface. This corresponds to a global selection of a preferred octonionic imaginary unit. The automorphisms leaving this selection invariant form group $SU(3)$ identifiable as color group. The selections of hyper-quaternionic sub-space under this condition are parameterized by CP_2. This means that each 4-surface in HO defines a 4-surface in $M^4 \times CP_2$ and one can speak about number-theoretic analog of spontaneous compactification having of course nothing to do with dynamics. It would be possible to make physics in two radically different geometric pictures: HO picture and $H = M^4 \times CP_2$ picture.

For a theoretical physicists of my generation it is easy to guess that the next step is to realize that it is possible to fix the preferred octonionic imaginary at each point of HO separately so that local $S^6 = G_2/SU(3)$, or equivalently the local group G_2 subject to $SU(3)$ gauge invariance, characterizes the possible choices of hyper-quaternionic structure with a preferred imaginary unit. $G_2 \subset SO(7)$ is the automorphism group of octonions, and appears also in M-theory. This local choice has interpretation as a fixing of the plane of non-physical

polarizations and rise to degeneracy which is a good candidate for the ground state degeneracy caused by the vacuum extremals.

$OH - -M^4 \times CP_2$ duality allows to construct a foliation of HO by hyper-quaternionic space-time surfaces in terms of maps $HO \rightarrow SU(3)$ satisfying certain integrability conditions guaranteing that the distribution of hyper-quaternionic planes integrates to a foliation by 4-surfaces. In fact, the freedom to fix the preferred imaginary unit locally extends the maps to $HO \rightarrow G_2$ reducing to maps $HO \rightarrow SU(3) \times S^6$ in the local trivialization of G_2. This foliation defines a four-parameter family of 4-surfaces in $M^4 \times CP_2$ for each local choice of the preferred imaginary unit. The dual of this foliation defines a 4-parameter family co-hyper-quaternionic space-time surfaces and it turns out that also these surfaces are needed.

Hyper-octonion analytic functions $HO \rightarrow HO$ with real Taylor coefficients provide a physically motivated ansatz satisfying the integrability conditions. The basic reason is that hyper-octonion analyticity is not plagued by the complications due to non-commutativity and non-associativity. Indeed, this notion results also if the product is Abelianized by assuming that different octonionic imaginary units multiply to zero. A good candidate for the HO dynamics is free massless Dirac action with Weyl condition for an octonion valued spinor field using octonionic representation of gamma matrices and coupled to the G_2 gauge potential defined by the tensor 7×7 tensor product of the imaginary parts of spinor fields.

The basic conjecture is that HQ and $coHQ$ surfaces correspond to preferred extremals of Kähler action. This conjecture has several variants. It could be that only asymptotic behavior corresponds to HQ analytic function but that HQ and $coHQ$ is a generic property. It could also be that maxima of Kähler function correspond to this kind of 4-surfaces. The encouraging hint is the fact that Hamilton-Jacobi coordinates appear naturally also in the construction of general solutions of field equations.

3. The notion of Kähler calibration

Calibration is a closed p-form, whose value for a given p-plane is not larger than its volume in the induced metric. What is important that if it is maximum for tangent planes of p-sub-manifold, minimal surface with smallest volume in its homology equivalence class results.

The idea of Kähler calibration is based on a simple observation. A hyper-octonionic spinor field defines a map $M^8 \rightarrow H = M^4 \times CP_2$ allowing to induce metric and Kähler form of H to M^8. Also Kähler action is well defined for the local hyper-quaternion plane.

The idea is that the non-closed 4-form associated the wedge product of unit tangent vectors of HQ plane in M^8 and saturating to volume for it becomes closed by multiplication with Kähler action density L_K. If L_K is minimal for hyper-quaternion plane, hyper-quaternionic manifolds define extremals of Kähler action for which the magnitudes of positive and negative contributions to the action are separately minimized.

In $coHQ$ case dual of the Kähler calibration results. In this case L_K would be most naturally maximal for HQ normal plane. There is also an alternative option but it is not favored by physical considerations.

This variational principle is not equivalent with the absolute minimization of Kähler action. Rather, in HQ case Universe would do its best to save energy, being as near as possible to vacuum. Also vacuum extremals would become physically relevant (they carry non-vanishing density gravitational energy). The non-determinism of the vacuum extremals would have an interpretation in terms of the ability of Universe to engineer itself. The attractiveness of the number theoretical variational principle from the point of calculability of TGD would be that the initial values for the time derivatives of the imbedding space coordinates at X^3 at light-like 7-D causal determinant could be computed by requiring that the energy of the solution is minimized. This could mean a computerizable construction of Kähler function.

In $coHQ$ phase Universe would obviously maximize fluctuations and contrasts in accordance with quantum criticality. One might say that these two phases give Universe kind of hawk-dove polarity.

One can assign to a given 3-surface both HQ and cHQ 4-surface in the generic case and the equivalence of descriptions requires that corresponding Kähler functions differ by the real part of a holomorphic function of CH coordinates.

4. Generalizing the notion of $HO - H$ duality to quantum level

The obvious question is how the $HO-H$ duality could generalize to quantum level. Number theoretical considerations combined with the general vision about generalized Feynman diagrams as a generalization of braid diagrams lead to general formulas for vertices in HO picture.

Simple arguments lead to the conclusion that strict duality can make sense only if the hyper-octonionic spinor field is second quantized in some sense. One can imagine two, not necessarily mutually exclusive, manners to quantize.

a) The construction of the spinor structure for the configuration space of 3-surfaces in HO forces to conclude that HO spinor fields induced to $X^4 \subset HO$ are second quantized as usual and define configuration space gamma matrices as super generators. The classical real-analytic HO spinor fields would represent analogs of zero modes of H spinor fields. The second quantized part of hyper-octonionic spinor fields induced to $X^4 \subset HO$ would have $1+1+3+\overline{3}$ decomposition having interpretation in terms of quarks and leptons and second quantized oscillator operators would commute with hyper-octonionic units. The detailed realization of $HO - H$ duality suggests that the induced spinor fields at $X^4 \subset H$ *resp.* $X^4 \subset HO$ are restrictions of H *resp.* HO spinor fields. This would hold for zero modes and could hold for second quantized part too.

b) The original idea was that the real Laurent coefficients correspond to a complete set of mutually commuting Hermitian operators having interpretation as observables. This is not enough for configuration space geometry but is favored by quantum classical correspondence. Space-time concept would be well defined only for the eigen states of these operators and physical states are mapped to space-time surfaces. The Hermitian operators would naturally correspond to the state space spanned by super Kac-Moody and super-canonical algebras, and quantum states would have precise space-time counterparts in accordance with quantum-classical correspondence.

The regions inside which the power series representing HO analytic function and matrix elements of G_2 rotation converge are identified as counterparts of maximal deterministic regions of the space-time surface. The Hermitian operators replacing Laurent coefficients are assumed to commute inside these regions identifiable also as coherence regions for the generalized Schrödinger amplitude represented by the HO spinor field.

By quantum classical correspondence these regions would be correlates for the final states of quantum jumps. The space-like 3-D causal determinants X^3 bounding adjacent regions of this kind represent quantum jumps. The hyper-octonionic part of the inner of the hyper-octonionic spinor fields at the two sides of the discontinuity defined as an integral over X^3 would give a number identifiable as complex number when imaginary unit is identified appropriately. The inner product would be identified as a representation of S-matrix element for an internal transition of particle represented by 3-surface, that is 2-vertex.

For the generalized Feynman diagrams n-vertex corresponds to a fusion of n 4-surfaces along their ends at X^3. 3-vertex can be represented number theoretically as a triality of three hyper-octonion spinors integrated over the 3-surface in question. Higher vertices can be defined as composite functions of triality with a map $(h_1, h_2) \rightarrow \overline{h}_3$ defined by octonionic triality and by duality given by the inner product. More concretely, $m+n$ vertex corresponds in HO picture to the inner product for the local hyper-octonionic products of m outgoing and n incoming hyper-octonionic spinor fields integrated over the 3-surface defining the vertex. Both 2-vertices representing internal transitions and $n > 2$ vertices are completely fixed. This should give some idea about the power of the number theoretical vision.

One can raise objections against the need for non-conventional quantization. The number theoretic prescription does not apply to the second quantized parts of HO spinor fields and S-matrix elements can be constructed using them so that two equivalent prescriptions of S-matrix would emerge. On the other hand, TGD inspired quantum measurement theory suggests dual codings S-matrix elements based on either quantum states or classical observables (zero modes) in 1-1 correspondence with them.

5. Does TGD reduce to 8-D WZW string model in HO picture?

Conservation laws suggests that in the case of non-vacuum extremals the dynamics of the local G_2 automorphism is dictated by field equations of some kind. The experience with WZW model suggests that in the case of non-vacuum extremals G_2 element could be written as a product $g = g_L(h)g_R^{-1}(\overline{h})$ of hyper-octonion analytic and anti-analytic complexified G_2 elements. g would be determined by the data at hyper-complex 2-surface for which the tangent space at a given point is spanned by real unit and preferred hyper-octonionic unit. Also Dirac action would be naturally restricted to this surface. The theory would reduce in HO picture to 8-D WZW string model both classically and quantally since vertices would reduce to integrals over 1-D curves. The interpretation of generalized Feynman diagrams in terms of generalized braid/ribbon diagrams and the unique properties of G_2 provide further support for this picture. In particular, G_2 is the lowest-dimensional Lie group allowing to realize full-powered topological quantum computation based on generalized braid diagrams and using the lowest level k=1 Kac Moody representation. Even if this reduction would occur only in special cases, such as asymptotic solutions for which Lorentz Kähler force vanishes or maxima of Kähler function, it would mean enormous simplification of the theory.

6. Why hyper-quaternionicity corresponds to the minimization of Kähler action?

The resulting over all picture leads also to a considerable understanding concerning the basic questions why (co)-hyper-quaternionic 4-surfaces define extrema of Kähler action and why WZW strings would provide a dual for the description using Kähler action. The answer boils down to the realization that the extrema of Kähler action minimize complexity, also algebraic complexity, in particular non-commutativity. A measure for non-commutativity with a fixed preferred hyper-octonionic imaginary unit is provided by the commutator of 3 and $\overline{3}$ parts of the hyper-octonion spinor field defining an antisymmetric tensor in color octet representation: very much like color gauge field.

Color action is a natural measure for the non-commutativity minimized when the tangent space algebra closes to complexified quaternionic, instead of complexified octonionic, algebra. On the other hand, Kähler action is nothing but color action for classical color gauge field defined by projections of color Killing vector fields. That WZW + Dirac action for hyper-octonionic strings would correspond to Kähler action would in turn be the TGD counterpart for the proposed string-YM dualities.

7. Various dualities and their physical counterparts

$HO - H$ duality is only one representative in a family of dualities characterizing TGD. It is not equivalent with $HQ - coHQ$ duality, which seems however to be equivalent with the electric-magnetic duality known for long. This duality relates descriptions based on space-like partonic 2-surfaces and time-like string orbits. $HO - H$ and $HQ - coHQ$ dualities seem to be closely correlated in the sense that HO picture is natural in HQ phase and H picture in $coHQ$ phase.

At configuration space level $HO - H$ duality means roughly following. In H picture spin and ew spin are spin-like quantum numbers whereas color is orbital quantum number and cannot be seen at space-time level directly. In HO picture the roles of these quantum numbers are changed. One could say that $HO-H$ duality acts as a super-symmetry permuting spin and orbital degrees of freedom of configuration space spinor fields. This duality allows a surprisingly detailed understanding of almost paradoxical dualities of hadron physics, and also explains proton spin crisis from first principles.

It seems possible to interpret $HO-H$ and $HQ-coHQ$ dualities as analogs of wave-particle duality in the infinite-dimensional context. For $HO - H$ duality the cotangent bundle of configuration space CH would be the unifying notion. Position q in CH would be represented by 3-surface whereas canonical momentum p would correspond to the same 3-surface but as a surface in CHO with induced metric and Kähler structure inherited from HO defining the tangent space of H. The notion of stringy configuration space might allow to understand also M-theory dualities in this manner.

1 Introduction

This chapter is second part of the multi-chapter devoted to the vision about TGD as a generalized number theory. The basic theme of the chapter is that TGD allows two dual pictures about space-time as a 4-surface. In the first picture space-times are regarded as hyper-quaternionic 4-surfaces in 8-dimensional hyper-octonionic space $HO = M^8$. In the second picture space-times are regarded as 4-surfaces in $M^4 \times CP_2$ satisfying field equations guaranteing absolute minimization of Kähler action.

1.1 Development of ideas

The discussions for years ago with Tony Smith [17] stimulated very general ideas about space-time surface as an associative, quaternionic sub-manifold of octonionic 8-space. Also the observation that quaternionic and octonionic primes have norm squared equal to prime in complete accordance with p-adic length scale hypothesis, led to suspect that the notion of primeness for quaternions, and perhaps even for octonions, might be fundamental for the formulation of quantum TGD. The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional quaternions and octonions.

It took some years to realize that the difficulties related to the realization of Lorentz invariance might be overcome by replacing quaternions and octonions with hyper-quaternions and hyper-octonions. Hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$ and can be regarded as a sub-space of complexified quaternions *resp.* octonions. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity. The loss of number field and even sub-algebra property is not fatal and has a clear physical meaning. The notion of primeness is inherited from that for complexified quaternions *resp.* octonions.

The notion of hyper-quaternionic and octonionic manifold makes sense but it not plausible that $H = M^4 \times CP_2$ could be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces are assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with a fixed complex structure (containing preferred imaginary unit) are labelled by CP_2, each hyper-quaternionic and co-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can loosely say that the number-theoretic analog of spontaneous compactification emerges: this of course has nothing to do with dynamics. Hyper-variants of number fields make also sense p-adically unlike the notions of number fields themselves. What deserves separate emphasis is that the basic structure of the standard model would reduced to number theory.

Some notational conventions are in order before continuing. The fields of quaternions *resp.* octonions having dimension 4 *resp.* 8 and will be denoted by Q and O. Their complexified variants will be denoted by Q_C and O_C. The sub-spaces of hyper-quaternions HQ and hyper-octonions HO are obtained by multiplying the quaternionic and octonionic imaginary units by $\sqrt{-1}$. These sub-spaces are very intimately related with the corresponding algebras, and can be seen as Euclidian and Minkowkian variants of the same basic structure. Also the Abelianized versions of the hyper-quaternionic and -octonionic sub-spaces can be considered: these algebras have a representation in the space of spinors of imbedding space $H = M^4 \times CP_2$.

1.2 Space-time-surface as a hyper-quaternionic or co-hyper-quaternionic sub-manifold of hyper-octonionic imbedding space?

Space-time identified as a hyper-quaternionic sub-manifold of the hyper-octonionic space in the sense that the tangent space of the space-time surface defines a hyper-quaternionic sub-space of the hyper-octonionic tangent space of H at each space-time point, looks an attractive idea. Also co-hyper-quaternionic surfaces correspond to space-time surfaces. Second possibility is that the algebra generated by tangent space *resp.* normal space of the space-time surface is associative (associativity *resp.* co-associativity). Also the possibility that the dynamics of the space-time surface is determined from the requirement that space-time surface X^4 is algebraically closed in the sense that tangential or normal algebra at each point corresponds to a 4-D sub-algebra of complexified octonions.

Some delicacies are caused by the question whether the induced algebra at X^4 is just the hyper-octonionic product or whether the algebra product is projected to the space-time surface. If the normal part of the product is projected out, the space-time algebra closes automatically.

The first guess would be that space-time surfaces are hyper-quaternionic of co-hyper-quaternionic sub-manifolds of hyper-octonionic space $HO = M^8$ with the property that complex structure is fixed and same at all points of space-time surface. This corresponds to a global selection of a preferred octonionic imaginary unit. The automorphisms leaving this selection invariant form group $SU(3)$ identifiable as color group. The selections of (co-)hyper-quaternionic sub-space under this condition are parameterized by CP_2. This means that each 4-surface in HO defines a 4-surface in $M^4 \times CP_2$ and one can speak about number-theoretic analog of spontaneous compactification having of course nothing to do with dynamics. It would be possible to make physics in two radically different geometric pictures: HO picture and $H = M^4 \times CP_2$ picture.

For a theoretical physicists of my generation it is easy to guess that the next step is to realize that it is possible to fix the preferred octonionic imaginary at each point of HO separately so that local $S^6 = G_2/SU(3)$, or equivalently the local group G_2 subject to $SU(3)$ gauge invariance, characterizes the possible choices of (co-)hyper-quaternionic structure with a preferred imaginary unit. $G_2 \subset SO(7)$ is the automorphism group of octonions, and appears also in M-theory. This local choice has interpretation as a fixing of the plane of non-physical polarizations and rise to degeneracy which is a good candidate for the ground state degeneracy caused by the vacuum extremals.

$OH - -M^4 \times CP_2$ duality allows to construct a foliation of HO by (co-)hyper-quaternionic space-time surfaces in terms of maps $HO \rightarrow SU(3)$ satisfying certain integrability conditions guaranteing that the distribution of (co-)hyper-quaternionic planes integrates to a foliation by 4-surfaces. In fact, the freedom to fix the preferred imaginary unit locally extends the maps to $HO \rightarrow G_2$ reducing to maps $HO \rightarrow SU(3) \times S^6$ in the local trivialization of G_2. This foliation defines a four-parameter family of 4-surfaces in $M^4 \times CP_2$ for each local choice of the preferred imaginary unit. The dual of this foliation defines a 4-parameter family co-hyper-quaternionic space-time surfaces. HQ and $coHQ$ surfaces intersect generically in a finite number of points.

Hyper-octonion analytic functions $HO \rightarrow HO$ with real Taylor coefficients provide a physically motivated ansatz satisfying the integrability conditions. The basic reason is that hyper-octonion analyticity is not plagued by the complications due to non-commutativity and non-associativity. Indeed, this notion results also if the product is Abelianized by assuming that different octonionic imaginary units multiply to zero. A good candidate for the HO dynamics is free massless Dirac action with Weyl condition for an octonion valued spinor field using octonionic representation of gamma matrices and coupled to the G_2 gauge potential defined by the tensor 7×7 tensor product of the imaginary parts of spinor fields.

The basic conjecture is that the absolute minima of Kähler action in $H = M^4 \times CP_2$ correspond to the hyper-quaternion analytic surfaces in HO. The map $f : HO \rightarrow S^6$ would probably satisfy some constraints posed by the requirement that the resulting surfaces define solutions of field equations in $M^4 \times CP_2$ picture. This conjecture has several variants. It could be that only the asymptotic behavior corresponds to hyper-quaternion analytic function but that hyper-quaternionicity is a general property of preferred extrema of Kähler action. The encouraging hint is the fact that Hamilton-Jacobi coordinates coding for the local selection of the plane of non-physical polarizations, appear naturally also in the construction of general solutions of field equations [D1].

It will be found that hyper-quaternion analytic surfaces cannot correspond to the absolute minima of Kähler action. Rather the absolute value of the contribution from a region with given sign of action density is either minimized or maximized. The most obvious guess is that HQ *resp.* $coHQ$ surfaces correspond to minima *resp.* maxima for the absolute values of these contributions. In particular, small deformations of empty Minkowski space *resp.* CP_2 type extremals would correspond to HQ *resp.* $coHQ$ surfaces. Also the dual Kähler action defined by the projection of CP_2 Kähler form to the normal space of space-time surface could be the action principle defining $coHQ$ 4-surfaces as its preferred extrema.

1.3 The notion of Kähler calibration

Calibration is a closed p-form, whose value for a given p-plane is not larger than its volume in the induced metric. What is important that if it is maximum for tangent planes of p-sub-manifold, minimal surface with smallest volume in its homology equivalence class results.

The idea of Kähler calibration is based on a simple observation. Hyper-octonionic spinor field defines a map $M^8 \rightarrow H = M^4 \times CP_2$ allowing to induce metric and Kähler form of H to M^8. Also Kähler action is well defined for the local hyper-quaternion plane.

The idea is that the non-closed 4-form associated the wedge product of unit tangent vectors of hyper-quaternionic plane in M^8 and saturating to volume for it becomes closed by multiplication with Kähler action density L_K. If L_K is minimal for hyper-quaternion plane, HQ manifolds define extremals of Kähler action for which the magnitudes of positive and negative contributions to the action are separately minimized. If L_K is maximal it could correspond $coHQ$ surfaces for which maximization occurs.

This variational principle is not equivalent with the absolute minimization of Kähler action. Rather, HQ Universe would do its best to save energy, being as near as possible to vacuum. Also vacuum extremals would become physically relevant (they carry non-vanishing density gravitational energy). The non-determinism of the vacuum extremals would have an interpretation in terms of the ability of Universe to engineer itself. The attractiveness of the number theoretical variational principle from the point of calculability of TGD would be that the initial values for the time derivatives of the imbedding space coordinates at X^3 at light-like 7-D causal determinant could be computed by requiring that the energy of the solution is minimized. This could mean a computerizable construction of Kähler function.

If maximization occurs in $coHQ$ case, contrasts defined by the values of Kähler action in regions of definite sign of action density are maximized in $coHQ$ phase. The obvious objection is that this option de-stabilizes the theory by implying large fluctuations but one might argue that quantum criticality requires this. This option does not necessarily imply energy maximization. For instance, CP_2 type extremals are vacuum extremals. If $coHQ$ corresponds to a minimization for the dual of Kähler action in the sense described, situation changes. For instance, CP_2 type extremals correspond to a vanishing dual of Kähler action density. It must be emphasized that HQ and $coHQ$ phases need not be dual to each other but could correspond to disjoint regions of configuration space of 3-surfaces.

1.4 Generalizing the notion of $HO - H$ duality to quantum level

The obvious question is how the $HO - H$ duality could generalize to quantum level. Number theoretical considerations combined with the general vision about generalized Feynman diagrams as a generalization of braid diagrams lead to general formulas for vertices in HO picture.

Simple arguments lead to the conclusion that strict duality can make sense only if the hyper-octonionic spinor field is second quantized in some sense. One can imagine two, not necessarily mutually exclusive, manners to quantize.

1. The construction of the spinor structure for the configuration space of 3-surfaces in HO forces to concluded that HO spinor fields induced to $X^4 \subset HO$ are second quantized as usual and define configuration space gamma matrices as super generators. The classical real-analytic HO spinor fields would represent analogs of zero modes of H spinor fields. The second quantized part of hyper-octonionic spinor fields induced to $X^4 \subset HO$ would have $1+1+3+\overline{3}$ decomposition having interpretation in terms of quarks and leptons and second quantized oscillator operators would commute with hyper-octonionic units. The detailed realization of $HO-H$ duality suggests that the induced spinor fields at $X^4 \subset H$ *resp.* $X^4 \subset HO$ are restrictions of H *resp.* HO spinor fields. This would hold for zero modes and could hold for second quantized part too.

2. The original idea was that the real Laurent coefficients correspond to a complete set of mutually commuting Hermitian operators having interpretation as observables. This is not enough for configuration space geometry but is favored by quantum classical correspondence. Space-time concept would be well defined only for the eigen states of these operators and physical states are mapped to space-time surfaces. The Hermitian operators would naturally correspond to the state space spanned by super Kac-Moody and super-canonical algebras, and quantum states would have precise space-time counterparts in accordance with quantum-classical correspondence.

The regions inside which the power series representing HO analytic function and matrix elements of G_2 rotation converge are identified as counterparts of maximal deterministic regions of the space-time surface. The Hermitian operators replacing Laurent coefficients are assumed to commute inside these regions identifiable also as coherence regions for the generalized Schrödinger amplitude represented by the HO spinor field.

By quantum classical correspondence these regions would be correlates for the final states of quantum jumps. The space-like 3-D causal determinants X^3 bounding adjacent regions of this kind represent quantum jumps. The hyper-octonionic part of the inner of the hyper-octonionic spinor fields at the two sides of the discontinuity defined as an integral over X^3 would give a number identifiable as complex number when imaginary unit is identified appropriately. The inner product would be identified as a representation of S-matrix element for an internal transition of particle represented by 3-surface, that is 2-vertex.

For the generalized Feynman diagrams n-vertex corresponds to a fusion of n 4-surfaces along their ends at X^3. 3-vertex can be represented number theoretically as a triality of three hyper-octonion spinors integrated over the 3-surface in question. Higher vertices can be defined as composite functions of triality with a map $(h_1, h_2) \rightarrow \overline{h}_3$ defined by octonionic triality and by duality given by the inner product. More concretely, $m+n$ vertex corresponds in HO picture to the inner product for the local hyper-octonionic products of m outgoing and n incoming hyper-octonionic spinor fields integrated over the 3-surface defining the vertex. Both 2-vertices representing internal transitions and $n > 2$ vertices are completely fixed. This should give some idea about the power of the number theoretical vision.

One can raise objections against the need for non-conventional quantization. The number theoretic prescription does not apply to the second quantized parts of HO spinor fields and S-matrix elements can be constructed using them so that two equivalent prescriptions of S-matrix would emerge. On the other hand, TGD inspired quantum measurement theory suggests dual codings S-matrix elements based on either quantum states or classical observables (zero modes) in 1-1 correspondence with them.

1.4.1 Does TGD in HO picture reduce to 8-D WZW string model?

Conservation laws suggests that in the case of non-vacuum extremals the dynamics of the local G_2 automorphism is dictated by field equations of some kind. The experience with WZW model suggests that in the case of non-vacuum extremals G_2 element could be written as a product $g = g_L(h)g_R^{-1}(\overline{h})$ of hyper-octonion analytic and anti-analytic complexified G_2 elements. g would be determined by the data at hyper-complex 2-surface for which the tangent space at a given point is spanned by real unit and preferred hyper-octonionic unit. Also Dirac action would be naturally restricted to this surface. The theory would reduce in HO picture to 8-D WZW string model like theory both classically and quantally. The duality of partonic H description with stringy HO description suggests that string orbits correspond to surfaces at which time like causal determinants intersect.

The interpretation of generalized Feynman diagrams in terms of generalized braid/ribbon diagrams and the unique properties of G_2 provide further support for this picture. In particular, G_2 is the lowest-dimensional Lie group allowing to realize full-powered topological quantum computation based on generalized braid diagrams and using the lowest level k=1 Kac Moody representation. Even if this reduction would occur only in special cases, such as asymptotic solutions for which Lorentz Kähler force vanishes or maxima of Kähler function, it would mean enormous simplification of the theory.

1.4.2 Why hyper-quaternionicity corresponds to the minimization of Kähler action?

The resulting over all picture leads also to a considerable understanding concerning the basic questions why hyper-quaternionic 4-surfaces define extrema of Kähler action and why WZW strings would provide a dual for the description using Kähler action. The answer boils down to the realization that the extrema of Kähler action minimize complexity, also algebraic complexity, in particular non-commutativity. A measure for non-commutativity with a fixed preferred hyper-octonionic imaginary unit is provided by the commutator of 3 and $\overline{3}$ parts of the hyper-octonion spinor field defining an antisymmetric tensor in color octet representation: very much like color gauge field. Color action is a natural measure for the non-commutativity minimized when the tangent space algebra closes to complexified quaternionic, instead of complexified octonionic, algebra. On the other hand, Kähler action is nothing but color action for classical color gauge field defined by projections of color Killing vector fields. That WZW + Dirac action for hyper-octonionic strings would correspond to Kähler action would in turn be the TGD counterpart for the proposed string-YM dualities. If the dual of Kähler action defines $coHQ$ 4-surfaces an analogous interpretation holds true.

1.4.3 Various dualities and their physical counterparts

$HO - H$ duality is only one representative in a family of dualities characterizing TGD. It is not equivalent with $HQ - coHQ$ duality, which seems however to be equivalent with the electric-magnetic duality known for long. This duality relates descriptions based on space-like partonic 2-surfaces and time-like string orbits. $HO - H$ and $HQ - coHQ$ dualities seem to be closely correlated in the sense that HO picture is natural in HQ phase and H picture in $coHQ$ phase.

At configuration space level $HO - H$ duality means roughly following. In H picture spin and ew spin are spin-like quantum numbers whereas color is orbital quantum number and cannot be seen at space-time level directly. In HO picture the roles of these quantum numbers are changed. One could say that $HO - H$ duality acts as a super-symmetry permuting spin and orbital degrees of freedom of configuration space spinor fields. This duality allows a surprisingly detailed understanding of almost paradoxical dualities of hadron physics, and also explains proton spin crisis from first principles.

It seems possible to interpret $HO - H$ and $HQ - coHQ$ dualities as analogs of wave-particle duality in the infinite-dimensional context. For $HO - H$ duality the cotangent bundle of configuration space CH would be the unifying notion. Position q in CH would be represented by 3-surface whereas canonical momentum p would correspond to the same 3-surface but as a surface in CHO with induced metric and Kähler structure inherited from HO defining the tangent space of H. The notion of stringy configuration space and the generalization of HQ-coHQ duality to dimension-codimension duality might allow to understand also M-theory dualities in this manner.

2 Quaternion and octonion structures and their hyper counterparts

In this section the notions of quaternion and octonion structures and their hyper counterparts are introduced with strong emphasis on the physical interpretation. Literature contains several variants of these structures (Hyper Kähler structure and quaternion Kähler structure [36]). The notion introduced here is inspired by the physical motivations coming from TGD and involves in an essential manner the notions of (hyper-)quaternion and (hyper-)octonion analyticity.

2.1 Motivations and basic ideas

Before going to details it is useful to make clear the constraints on the concept of the hyper-octonionic structure implied by TGD view about physics.

$M^4 \times CP_2$ cannot certainly be regarded as having any global octonionic structure (for instance in the sense that it could be regarded as a coset space associated with some exceptional group). There are however clear indications for the importance of the hyper-quaternionic and -octonionic structures.

1. $SU(3)$ is the only simple 8-dimensional Lie-group and acts as the group of isometries of CP_2: if $SU(3)$ had some kind of octonionic structure, CP_2 would become unique candidate for the space S. The decomposition $SU(3) = h + t$ to $U(2)$ subalgebra and its complement corresponds rather closely to the decomposition of (hyper-)octonions to (hyper-)quaternionic sub-space and its complement. The electro-weak $U(2)$ algebra has a natural 1+3 decomposition and generators allow natural hyper-quaternionic structure. Hyper Kähler structure with three covariantly constant quaternionic imaginary units represented Kähler forms is however not possible. The components of the Weyl tensor of CP_2 behave with respect to multiplication like quaternionic imaginary units but only one of them is covariantly constant so that hyper-Kähler structure is not possible.

2. M^4_+ has a natural 1+3 decomposition and a unique cosmic time coordinate defined as the light cone proper time. Hyper-quaternionic structure is consistent with the Minkowskian signature of the inner product and hyper quaternion units have a natural representation in terms of covariantly constant self-dual symplectic forms and their contractions with sigma matrices.

In the following only (hyper-)octonion structure is considered: the generalization to the (hyper-)quaternion case is trivial. One can imagine two approaches to the definition of (hyper)-octonion structure.

1. (Hyper-)octonionic manifolds are obtained by gluing together coordinate patches using (hyper-)octonion analytic functions with real Laurent coefficients (this guarantees associativity and commutativity). This definition does not yet involve metric or any other structures (such

as Kähler structure). This structure seems to be a necessary ingredient of any definition confirming in spirit with TGD.

2. If the manifold is endowed with metric, octonionic structure should be defined as a local tangent space structure analogous to eight-bein structure and local gauge algebra structures. This can be achieved by contracting octo-bein vectors with the standard octonionic basis to get octonion form I_k. Each vector field a^k defines naturally octonion field $A = a^k I_k$. The product of two vector fields can be defined by the octonionic multiplication and this leads to the introduction of a tensor field d_{klm} of these structure constants obtained as the contraction of the octobein vectors with the octonionic structure constants d_{abc}. Hyper-octonion structure can defined in a completely analogous manner.

A possibly relevant notion is the induction of (hyper-)octonion structure.

1. It is possible to induce octonionic structure to any 4-dimensional space-time surface by forming the projection of I_k to the space-time surface and redefining the products of I_k:s by dropping away that part of the product, which is orthogonal to the space-time surface. This means that the structure constants of the new 4-dimensional algebra are the projections of d_{klm} to the space-time surface. One can also define similar induced algebra in the 4-dimensional normal space of the space-time surface.

2. The projection is not absolutely necessary and its is possible to have quaternionic associative tangent spaces without this assumption. As a matter fact, this option seems to be the physically favored one, and leads naturally to the hyper-quaternionicity constraint on space-time surfaces. An attractive hypothesis is that the induced tangential or normal space algebra is associative or hyper-quaternionic algebra. Also co-associativity defined as associativity of the normal space algebra is possible. This property would give for the 4-dimensionality of the space-time surface quite special algebraic meaning.

2.2 Octonions and quaternions

In the following only the basic definitions relating to octonions and quaterions are given. There is an excellent article by John Baez [29] describing octonions and their relations to the rest of mathematics and physics.

Octonions can be expressed as real linear combinations $\sum_k x^k I_k$ of the octonionic real unit $I_0 = 1$ (counterpart of the unit matrix) and imaginary units I_a, $a = 1, ..., 7$ satisfying

$$\begin{aligned} I_0^2 &= I_0 \equiv 1 \ , \\ I_a^2 &= -I_0 = -1 \ , \\ I_0 I_a &= I_a \ . \end{aligned} \tag{1}$$

Octonions are closed with respect to the ordinary sum of the 8-dimensional vector space and with respect to the octonionic multiplication, which is neither commutative ($ab \neq ba$ in general) nor associative ($a(bc) \neq (ab)c$ in general).

A concise manner to summarize octonionic multiplication is by using octonionic triangle. Each line (6 altogether) containing 3 octonionic imaginary units forms an associative triple which together with $I_0 = 1$ generate a division algebra of quaternions. Also the circle spanned by the 3 imaginary units at the middle of the sides of the triangle is associative triple. The multiplication rules for each associative triple are simple:

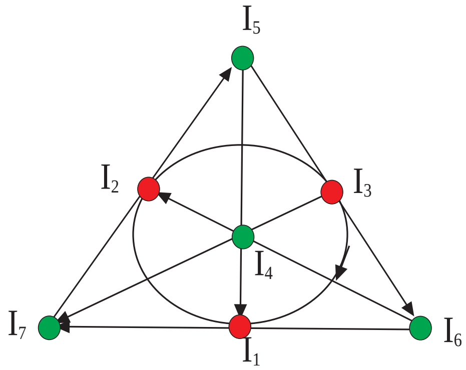

Figure 1: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

$$I_aI_b \quad = \quad \epsilon_{abc}I_c \quad , \tag{2}$$

where ϵ_{abc} is 3-dimensional permutation symbol. $\epsilon_{abc}=1$ for the clockwise sequence of vertices (the direction of the arrow along the circumference of the triangle and circle). As a special case this rule gives the multiplication table of quaternions. A crucial observation for what follows is that any pair of imaginary units belongs to one associative triple.

The non-vanishing structure constants ${d_{ab}}^c$ of the octonionic algebra can be read directly from the octonionic triangle. For a given pair I_a, I_b one has

$$\begin{aligned}
I_aI_b &= {d_{ab}}^cI_c \quad , \\
d_{ab} \quad c &= {\epsilon_{ab}}^c \quad , \\
I_a^2 &= {d_{aa}}^0I_0 = -I_0 \quad , \\
I_0^2 &= {d_{00}}^0I_0 \quad , \\
I_0I_a &= {d_{0a}}^aI_a = I_a \quad .
\end{aligned} \tag{3}$$

For ϵ_{abc} c belongs to the same associative triple as ab.

Non-associativity means that is not possible to represent octonions as matrices since matrix product is associative. Quaternions can be represented and the structure constants provide the defining representation as $I_a \rightarrow d_{abc}$, where b and c are regarded as matrix indices of 4×4 matrix. The algebra automorphisms of octonions form 14-dimensional group G_2, one of the so called exceptional Lie-groups. The isotropy group of imaginary octonion unit is the group $SU(3)$. The Euclidian inner product of the two octonions is defined as the real part of the product $\overline{x}y$

$$\begin{aligned}
(x,y) &= Re(\overline{x}y) = \sum_{k=0,..7} x_ky_k \quad , \\
\overline{x} &= x^0I_0 - \sum_{i=1,..,7} x^kI_k \quad ,
\end{aligned} \tag{4}$$

and is just the Euclidian norm of the 8-dimensional space.

2.3 Hyper-octonions and hyper-quaternions

The Euclidicity of the quaternion norm suggests that octonions are not a sensible concept in TGD context. One can imagine two manners to circumvent this conclusion.

1. M^4 metric as real part of product...

Minkowskian metric for octonions and quaternions is obtained by identifying Minkowski inner product xy as the real counterpart of the product

$$x \cdot y \equiv Re(xy) = x^0y^0 - \sum_k x^ky^k \ . \tag{5}$$

$SO(1,7)$ ($SO(1,3)$ in quaternionic case) Lorentz invariance appears completely naturally as the symmetry of the real part of the octonion (quaternion) product and hence of octonions/quaternions and there is no need to perform the complexification of the octonion algebra. Furthermore, only the signature $(1,7)$ ($(1,3)$ in the quaternionic case) is possible and this would raise $M^4_+ \times CP_2$ in a preferred position.

This norm does not give rise to a number theoretic norm defining a homomorphism to real numbers. Indeed, the number theoretic norm defined by the determinant of the linear mapping defined by the multiplication with quaternion or octonion, is inherently Euclidian. This is in conflict with the idea that quaternionic and octonionic primes and their infinite variants should have key role in TGD [E3].

2. ....or hyper-octonions and -quaternions?

Hyper-octonions and hyper-quaternions provide a possible solution to these problems. These are obtained by multiplying imaginary units by $\sqrt{-1}$. These numbers form a sub-space of complexified octonions/quaternions and the cross product of imaginary parts leads out from this sub-space. In this case number theoretic norm induced from Q_C/O_C gives the fourth/eighth power of Minkowski length and Lorentz group acts as its symmetries. Light-like hyper-quaternions and -octonions causing the failure of the number field property have also a clear physical interpretation.

A criticism against the notion of hyper-quaternionic and octonionic primeness is that the tangent space as an algebra property is lost and the notion of primeness is inherited from Q_C/O_C. Also non-commutativity and non-associativity could cause difficulties. The proposed representation of hyper-quaternionic sub-manifolds in terms of real-analytic hyper-octonion analytic maps is equivalent with the the version based on maps using Abelian version of hyper-octonions for which the products of different imaginary units give zero. This observation allows to understand why the potential difficulties associated with non-commutativity and non-associativity can be circumvented.

2.4 p-Adic length scale hypothesis and quaternionic and hyper-quaternionic primes

p-Adic length scale hypothesis [E5] states that fundamental length scales correspond to the p-adic length scales proportional to $\sqrt{p}$, p prime. Even more: the p-adic primes $p \simeq 2^k$, k prime or possibly power of prime, are especially interesting physically. The so called elementary particle-blackhole analogy gives a partial theoretical justification for this hypothesis [E5]. A strong empirical support for the hypothesis comes from p-adic mass calculations [F2, F3, F4, F5].

Elementary particles correspond to the so called CP_2 type extremals in TGD Universe [D1, E5]. Elementary particle horizon can be defined as a surface at which the Euclidian signature of the metric of the space-time surface containing topologically condensed CP_2 type extremal, changes to Minkowskian signature. The generalization of the Hawking-Bekenstein formula relates the real counterpart of the p-adic entropy associated with the elementary particle to the area of the elementary particle horizon. If one requires that the radius of the elementary particle horizon corresponds to a p-adic length scale: $R = L(k)$ or $k^{n/2}L(k)$ where k is prime, then p is automatically near to 2^{k^n} and p-adic length scale hypothesis is reproduced! The proportionality of length scale to $\sqrt{p}$, rather than p, follows from p-adic thermodynamics for mass squared (!) operator and from Uncertainty Principle.

What Tony Smith [17] suggested, was a beautiful connection with number theory based on the generalization of the concept of a prime number. In the so called D^4 lattice regarded as consisting of integer quaternions, one could identify prime quaternions as the generators of the multiplicative algebra of the integer quaternions. From the basic properties of the quaternion norm it follows directly that prime quaternions correspond to the 3-dimensional spheres $R^2 = p$, p prime, with integer value E^4 coordinates. The worries are of course raised by the Euclidian signature of the number theoretical norm of quaternions.

2.4.1 Hyper-quaternionic and -octonionic primes and effective 2-dimensionality

The notion of prime generalizes to hyper-quaternionic and -octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic primes can be represented as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p-1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$. The interpretation of hyper-quaternionic primes (or integers) as four-momenta suggests itself. Note that it is not possible to find a rest system for a massive particle unless the energy is allowed to be a square root of integer.

The notion of "irreducible" (see Appendix of [E1]) is defined as the equivalence class of primes related by a multiplication with a unit (integer with unit norm) and is more fundamental than that of prime. All Lorentz boosts of a hyper prime obtained by multiplication with units labelling $SO(D-1)$ cosets of $SO(D-1,1)$, $D = 4, 8$ to a hyper prime, combine to form a hyper irreducible. Note that the units cannot correspond to real particles in the arithmetic quantum field theory in which primes correspond to D-momenta labelling the physical states.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the hyper-complex case. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality.

Hyper-complex numbers H_2 define the maximal sub-algebra of HQ and HO. In the case of H_2 the failure of the number field property is due to the existence of light-like hyper-complex numbers with vanishing norm. The light-likeness of hyper-quaternions and -octonions is expected to have a deep physical significance and could define a number theoretic analog of propagator pole and light-like 3-D and 7-D causal determinants.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes. Note that the hyper-octonionic primes related by $SO(7,1)$ boosts need not represent physically equivalent states.

The situation becomes more complex if also space-like hyper primes with negative norm squared $n_0^2 - n_1^2 - ... = -p$ are allowed. Gaussian primes with $p \ mod4 \ = 1$ would be representable as primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$. If all quaternionic primes allow a representation as a quaternionic integer with three non-vanishing components, they can be identified as space-like hyper-quaternionic primes. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers. By their tachyonity space-like primes are not physically favored.

2.4.2 Hyper-quaternionic hyperboloids and p-adic length scale hypothesis

In the hyper-quaternionic case the 3-dimensional sphere $R^2 = p$ is replaced with Lobatchevski space (hyperboloid of M^4 with points having integer valued M^4 coordinates. Hence integer valued hyper-quaternions allow interpretation as quantized four-momenta.

Prime mass hyperboloids correspond to $n = p$. It is not possible to multiply hyperboloids since the cross product leads out of hyper sub-space. It is however possible to multiply the 2-dimensional hyperboloids and act on these by units to get full 3-D hyperboloids. The powers of hyperboloid p correspond to a multiplicatively closed structure consisting of powers p^n of the hyperboloid p. At space-time level the hyper-quaternionic lattice gives rise to a one-dimensional lattices of hyperboloidal lattices labelled by powers p^n, and the values of light-cone proper time $a \propto \sqrt{p}$ are expected to define fundamental p-adic time scales.

Also the space-like hyperboloids $R^2 = -n$ are possible and the notion of primeness makes sense also in this case. The space-like hyperboloids define one-dimensional lattice of space-like hyper-quaternionic lattices and an explanation for the spatial variant of the p-adic length scale hypothesis stating that p-adic length scales are proportional to $\sqrt{p}$ emerges in this manner naturally.

2.4.3 Euclidian version of the p-adic length scale hypothesis

Hyper-octonionic integers have a decomposition into hyper-quaternion and a product of $\sqrt{-1}K$ with quaternion so that quaternionic primes can be identified as hyper-octonionic space-like primes. The Euclidian version of the p-adic length scale hypothesis follows if one assumes that the Euclidian piece of the space-time surrounding the topologically condensed CP_2 type extremal can be approximated with a quaternion integer lattice with radius squared equal to $r^2 = k^n$, k prime. One manner to understand the finiteness in the time direction is that topological sum contacts of CP_2 type extremal are not static 3-dimensional topological sum contacts but genuinely four-dimensional: 3-dimensional contact is created, expands to a maximum size and is gradually reduced to point. The Euclidian space-time volume containing the contact would correspond to an Euclidian region $R^2 = k^n$ of space-time. The distances of the lattice points would be measured using the induced metric. These contacts could have arbitrarily long duration from the point of view of external observer since classical gravitational fields give rise to strong time dilation effects (strongest on the boundary of the Euclidian region where the metric becomes degenerate with the emergence of a light like direction).

Lattice structure is essential for the argument. Lattice structures of type D^4 indeed emerge naturally in the construction of the p-adic counterparts of the space-time surfaces as p-adically analytic surfaces. The essential idea is to construct the p-adic surface by first discretizing space-time surface using a p-adic cutoff in k:th pinary digit and mapping this surface to its p-adic counterpart and complete this to a unique smooth p-adically analytic surface.

This leads to a fractal construction in which a given interval is decomposed to p smaller intervals, when the resolution is increased. In the 4-dimensional case one naturally obtains a fractal hierarchy of nested D^4 lattices. The interior of the elementary particle horizon with Euclidian signature corresponds to some subset of the quaternionic integer lattice D^4: an attractive possibility is that the absolute minimization of the Kähler action and the maximization of the Kähler function force this set to be a ball $R^2 \leq k^n$, k prime.

2.5 Manifolds with (hyper-)octonion and (hyper-)quaternion structure

The definition of the notions of (hyper-)octonionic and (hyper-)quaternionic manifolds is straightforward. Since vielbein structure determines the geometry of the imbedding space completely, it seems natural to relate (hyper-)octonionic structure to the vielbein structure so that (hyper-)octonion structure becomes essentially metric concept. In the following only the Minkowskian case is considered in detail with restriction to hyper-quaternionic/octonionic case.

2.5.1 The notion of hyper-quaternionic/-octonionic analyticity

The crucial observation is that hyper-analytic series with real coefficients does not lead out from the hyper-subspace. Hence coordinate atlases based on hyper-analytic coordinate maps are possible and the notions of hyper-quaternionic and -octonionic manifolds are well-defined.

Since cross product terms in the (hyper-)octonionic Laurent series with real coefficients vanish, the real-analytic (hyper-)quaternionic and (hyper)-octonionic power series are expressible as

$$h_0 + \overline{h} \quad \rightarrow \quad ah_0 + b\overline{h} \ , \tag{6}$$

where the coefficients a and b depend only on h_0 and $|\overline{h}|^2$. This means that the result is linear in the imaginary part of h and in this case non-commutativity and non-associativity do not cause difficulties in the definition of derivatives. Hence the notion (hyper-)octonionic analytic map of HO to itself is well-defined and the notion of (hyper-)octonionic manifold makes sense since coordinate maps relating different coordinate patches can be (hyper-)quaternionic.

A more general HQ/HO analytic map results by allowing a global rotation of $\overline{h}$ induced by an automorphism of (hyper-)quaternions or (hyper-)octonions. Since a and b depend on automorphism invariants only, these automorphisms commute with HQ/HO analytic maps. Even more general notion of hyper-analyticity results when this rotation is allowed to be local.

The sub-group of the automorphism group $G_2 \subset SO(7)$ of octonions leaving a given imaginary octonion unit, say e_7 invariant, is $SU(3)$ and with respect to this group octonions decompose to two color singlets plus triplet and anti-triplet. The tensor product of triplets gives rise to a color octet defining an element of $SU(3)$ Lie algebra playing a crucial role in the proposed representation of space-time surfaces as hyper-quaternionic 4-surfaces of HO defined by hyper-octonion analytic maps.

2.5.2 Metric and vielbein

The ordinary inner product $Re(x\overline{y})$ can be used with conjugation acting on the hyper-octonionic/-quaternionic imaginary units but leaving $\sqrt{-1}$ invariant. This inner product can be lifted to the ordinary inner product for vector fields expressible as $a = a^k I_k$ in terms of the hyper vector fields related to the standard hyper basis I_a by a multiplication with hyper vielbein' e_k^a,

$$I_k \quad = \quad e_k^a I_a \ , \tag{7}$$

where I_a, $a \neq 0$, is multiplied with $\sqrt{-1}$ in hyper-case. Each local vielbein $SO(D-1,1)$ rotation gives rise to a new basis at each point of the M^D $(D = 4, 8)$ but respects hyper inner product. Hence one can say that hyper structure is consistent with local $SO(D-1,1)$ gauge invariance.

One cannot perform arbitrary vierbein rotations of the quaternion units as is clear from the fact that I_0, which appears in a special role in the inner product, must be invariant under the automorphisms. In the case of the (hyper-)quaternions the automorphism group is $SO(3)$. In the case of the future light cone, the invariance of I_0 is natural if it corresponds to the Lorentz invariant proper time coordinate. In the case of hyper-octonions the allowed transformations must respect octonionic multiplication table and correspond to the group G_2.

2.5.3 The notions of (hyper-)octonion and (-)quaternion Hermitian manifolds

The notion of Hermitian metric is a crucial element of conformal invariance and it would be highly desirable to generalize this notion. The generalization of the notion of Hermitian metric forces naturally the selection of preferred quaternionic and complex planes in a manifold possessing octonion Hermitian structure.

1. Quaternionic case

For quaternions the line element can be expressed a a bilinear $dqd\overline{q}$. Thus q and its Hermitian conjugate resulting as anti-automorph define the first pair of coordinates. In order to obtain the second pair, the introduction of a preferred imaginary unit, call it e_1, is needed. The automorphic conjugate $q_1 = q_0 - q_1 e_1 + q_2 e_2 - q_3 e_1 e_2$ and its Hermitian conjugate define the second coordinate pair, and the line element can be expressed as

$$ds^2 = \frac{1}{2}\left[dqd\overline{q} + dq_1 d\overline{q}_1\right] \ .$$

The first guess is that for a general 4-manifold with quaternion Hermitian structure the generalization of the metric would read as

$$ds^2 = F dqd\overline{q} + G dq_1 d\overline{q}_1 \ .$$

Here F and G are functions of quaternion coordinates. The requirement that real quaternion analyticity provides a general solution to the Laplacian equation

$$\partial_\alpha (g^{\alpha\beta} g^{1/4} \partial\beta)\Psi = 0 \tag{8}$$

associated with a half density (spinor field most naturally) requires that the metric disappears from the equation. This implies a stronger condition

$$ds^2 = F\left[dqd\overline{q} + dq_1 d\overline{q}_1\right] \ . \tag{9}$$

The condition is so strong that space-time surfaces in $M^4 \times CP_2$ are not expected to satisfy it. The condition might however hold true for the hyper-quaternionic 4-surfaces of HO.

Real-analytic quaternion transformations are expected to induce a mere scaling of the metric determinant. For a general manifold with quaternion Hermitian structure the choice of the complex sub-space of the tangent space of quaternions is expected to depend on the point of the manifold and defines a map from the manifold to the sphere S^2 labelling the complex tangent planes of Q. The argument generalizes in a trivial manner to the case of HQ. In this case a $SO(3)$ connection is needed in order to define the parallel translation.

2. Octonionic case

In the octonionic case quaternionic sub-space of octonions is needed in order to define the Hermitian structure. The four automorphic quaternion conjugates induce three automorphic conjugates o_i, $i = 2, 3, 4$ of the octonion variable $o_1 = q_1 + e_3 q_2$. The variables o_i and their octonionic Hermitian conjugates define 8 octonionic variables. The line element of octonionic manifold in the general case has the same form as in the quaternionic case. Half densities as natural real-analytic solutions of Laplace equation are replaced with 1/4-densities in this case.

In the general case the local quaternionic tangent sub-space depends on the point of the octonionic manifold. Hence the introduction of octonion Hermitian structure automatically forces the selection of a local quaternion sub-space and the Hermitian structure for the latter forces the selection of a local complex sub-space.

These considerations generalize in a trivial manner to the hyper-octonionic case. The generalization of the concept of Hermiticity provides support for the idea that HO is foliated by space-time surfaces defined by an integrable distribution of hyper-quaternionic planes of the tangent space of HO. Also the local selection of the preferred imaginary unit emerges naturally if the space-time surfaces are required to have a quaternion Hermitian structure.

2.5.4 Can one regard CP_2 and M^4_+ as Euclidian and Minkowskian variants of hyper-quaternionic projective space?

The notion of projective space generalizes also to the hyper-quaternionic case and one can ask whether it is possible to interpret future light-cone M^4_+ and CP_2 as hyper-quaternionic projective spaces.

The points of a 1-dimensional hyper-quaternionic projective space HP_1 would be pairs of points (h_1, h_2) with the equivalence relation $(h_1, h_2) \equiv \lambda(h_1, h_2)$, $\lambda \neq 0$. The two projective coordinate charts can be defined in the standard manner as $(h_a = h_1/h_2, 1)$ or as $(1, h_b = h_2/h_1)$. The generalization to the case of HP_n is obvious.

In the case of hyper-quaternionic numbers the failure of the number field property implies that the coordinate singularities corresponding to $q_1 = 0$ *resp.* $q_2 = 0$ are replaced by coordinate singularities corresponding to all light-like values of h_1 *resp.* h_2. Thus the space in question can be interpreted as the intersection of future and past light-cones. The boundaries of the cones intersect at points where both h_1 and h_2 are light-like. This brings in mind the the fact that S-matrix involves in the minimal situation future and past directed light-cones with partonic 2-surfaces representing incoming and outgoing particles located at the boundaries of these light-cones.

This observation supports the view that $M^4_+(a_1) \cap M^4_-(a_2)$ and CP_2 emerge naturally as Minkowskian and Euclidian variants of the hyper-quaternionic projective space.

1. If the metric of the hyper-quaternionic projective space has Minkowskian signature then the natural identification of HP_1 is as $M^4_+(a_1) \cap M^4_-(a_2)$. The boundary of HP_1 is metrically 2-dimensional but topologically 3-dimensional. Light-cone boundary is hyper-quaternionic space itself since scalings respect the light-likeness of the projective coordinates. It is possible to construct several projective spaces by posing conditions on projective scalings such as $\lambda_0 > 0$ and selecting regions of M^4 properly by posing conditions on the sign of M^4 time coordinate. For instance, M^4 with light-cone boundary excluded is possible and becomes full M^4 when the boundary is added.

2. If the metric of the hyper-quaternionic projective space has an Euclidian signature, metric 2-dimensionality requires topological 2-dimensionality, and it is necessary to identify the points having different values of the light-like radial coordinate and the boundary becomes sphere S^2 attached to E^4. The resulting space would be nothing but CP_2. Thus CP_2 and M^4 are very closely related.

One can of course argue that Euclidian signature means that hyper-quaternions are replaced by quaternions. It is indeed known that CP_2 allows quaternion Kähler structure [36] which is weaker structure than Hyper Kähler structure. Even in Kähler metric making CP_2 symmetric space the components of Weyl tensor obey quaternionic multiplication table but only one component of the Weyl tensor is covariantly constant. In fact, the breaking of the quaternion structure to a unique complex structure is what extends holonomy group from $SU(2)$ forced by the Hyper Kähler structure to $U(2)$ and brings in the missing $U(1)$ factor of the electro-weak gauge group. The result would mean that $M^4 \times CP_2$ can be regarded as product of hyper-quaternionic and quaternion Kähler manifolds.

The key question is whether $M^4 \times CP_2$ could be regarded as hyper-octonionic manifold in some sense. It is highly improbable that the topology of $M^4 \times CP_2$ would allow HO-analytic coordinate maps between different coordinate patches since complex analytic coordinate maps allow much more structure than HO-analytic coordinate maps. The basic 8-dimensional hyper-octonionic spaces are just HO and the corresponding projective space PHO and variants of this space.

2.6 Light-like causal determinants, number theoretic light-likeness, and generalization of residue calculus

The poles and cuts of complex functions correspond in hyper-quaternionic *resp.* octonionic framework to 3- *resp.* 7-dimensional surfaces at which hyper-quaternionic *resp.* hyper-octonionic variable is light-like. This raises obvious questions. How the number-theoretic light-likeness in HO relates to the metric light-likeness in $M^4 \times CP_2$? Does the residue calculus generalize to the hyper analytic context and provide a generalization of the basic formulas of conformal field theory?

2.6.1 Is there a relationship between metric light-likeness and hyper-quaternionic light-likeness?

In the case of $HQ = M^4$ and $HO = M^8$ the metric light-cones correspond to the light-likeness of the hyper counterpart h of Minkowski coordinate. For HQ- and HO-analytic functions the image of point h is given by $h = ah_0 + bO_h(\overline{h})$, where O_h corresponds to a local $G_2 \subset SO(7)$ rotation, and a and b are $SO(7)$ invariants. Light-likeness condition reads as $a^2h_0^2 - b^2|\overline{h}|^2 = 0$. The question is whether this condition could correspond to the metric light-likeness in the metric induced from Minkowski metric. For the map $w = h^2$ the light-likeness corresponds to that for h and thus to light-cone as is easy to see. By the multiplicative property of the number theoretical norm this is the case also for h^n and for any real-analytic power series which vanishes at $h = 0$. Thus HQ and HO hyper-analytic map seem to respect causality in a well-defined sense.

This and the central role of 3-D and 7-D light like causal determinants in the formulation of quantum TGD inspire some questions.

1. Could the number theoretic light-likeness in HQ and HO quite generally correspond to metric light-likeness in the induced metric.
2. Could the metric light-likeness of 3-D causal determinants $X_l^3 \subset X^4 \subset M^4 \times CP_2$ in the induced metric be equivalent with the light-likeness with respect to the metric induced from OH. This would be a natural condition on the correspondence between HO and $M^4 \times CP_2$ representations of the X^4.
3. Is the hyper-quaternionic counterpart of Kähler structure possible. In other words, does the metric of space-time surface induced from HO possess only non-diagonal components in hyper-quaternionic coordinates? If this were the case, hyper-quaternion analytic transformations of $X^4 \subset HO$ would induce an analog of conformal scaling of the metric determinant, and could be interpreted as active transformations of space-time surface modifying its shape. Metric determinant of HQ Hermitian metric would transformed by the hyper-quaternionic norm of df/dh to the product of its all conjugates. Thus these map would preserve the character of light-like causal determinants with $\sqrt{g} = 0$.

2.6.2 Singularities of hyper analytic maps

In ordinary complex analysis the singularities of analytic maps are important. The map $z \rightarrow w = \sqrt{z}$ is the basic example. It creates two-fold covering of complex plane having singularity at origin. The hyper-elliptic Riemann surfaces in C^2 provide a more interesting example: in this case double covering of D^2 is in question except in points which correspond to degenerate roots of second degree polynomial. The singularities of hyper-quaternion analytic maps $h \rightarrow f(h)$ are expected to correspond to the light-likeness of df/dh.

Hyper-quaternionic 4-surfaces of HO with coordinate $H = h_1 + e_3 h_2$ are represented as solutions of system of form $F_i(h_1 + e_3 h_2) = 0$, $i = 1, ..., 4$. This gives $h_2 = f(h_1)$ and h_2. One might hope that f is hyper-quaternion analytic function with real Laurent coefficients. This function is in general multi-valued and when some roots co-inside $df/dh_1 = 0$ holds true. By $df/dh_1 = -(dF/dh_1)/(dF/dh_2)$ this corresponds to the vanishing of either dF/dh_1 or dF/dh_2 and to discrete points of the space-time surface. Something singular would happens also at the 3-D surfaces at which dF/dh_1 or dF/dh_2 is light-like.

2.6.3 Does the hyper variant of the residue calculus exist?

Residue calculus is in a key role in the complex analysis and thus in the formulation of conformal field theories. One might wonder whether its generalization to (hyper-)quaternionic and (-)octonionic case might exist and be useful in quantum TGD context. The fact that hyper-quaternion/-octonion analytic functions with real Laurent coefficients are linear in the imaginary part $\overline{h}$ of the argument implies effective commutativity and associativity and could make the notion of integral function and even definite integral well defined.

As a matter fact, the same notion of analyticity results if it is assumed that quaternionic units annihilate each other as in the induced Abelian algebra obtained by regarding hyper-quaternions as sub-space of complexified quaternions and projecting normal component from the product.

The physical intuition serves as a guideline in attempts to guess what the generalization of integrals $\int f(z)dz$ over curves of complex plane might mean.

The construction of configuration space geometry and of physical states reduces to the data given at two-dimensional partonic surfaces, which have co-dimension two as have also the poles of an analytic function. The hyper-quaternionic counterparts of residue integrals correspond to integrals over codimension 1 surfaces X^3 in X^4. Thus it would seem that 3-D light-like causal determinants are more like cuts than poles. These integrals should reduce to integrals over partonic two-surfaces X^2 defined by the intersections $X^3 \cap X^3_l$, perhaps defined by the value of the integrand at these surfaces serving as end points of integration curve.

A good guess is that admissible integration paths X^3 correspond to light-like 3-surfaces X^3_l having interpretation as lines of generalized Feynman diagrams. By taking one integration variable to be h they would reduce to sum of 2-dimensional integrals over partonic 2-surfaces X^2. Hyper-quaternion analyticity requires that the determinant of the induced metric, which is certainly non-analytic function, does not appear in the admissible integrands. Hence these integrals could define conformal (or hyper-conformal) invariants. These kind of invariants would naturally appear in the definition of S-matrix elements using generalized Feynman diagrams for which by definition diagrams with loops are equivalent to tree diagrams.

Let us see whether these ideas survive more quantitative inspection. For hyper-quaternionic function $1/h$ in $HQ = M^4$ 3-dimensional light-cone $t^2 - x^2 - y^2 - z^2$ defines the singularity, and could be also seen as the analog of a cut rather than pole of an analytic function. For $HO = M^8$ 7-dimensional light-like cone takes the same role.

The idea can be tested in the case of H_2 by calculating the integral $\int dh/h$ around closed curve intersecting light-cone $a^2 = t^2 - z^2 = 0$ twice. The integral function is $log(h)$, with $h = \pm\sqrt{(}|a^2|)exp(e_1\eta)$ using the hyperbolic analog of polar coordinates. The modulus of h has now both signs and is discontinuous along the 2-D light-cone boundary. The integral reduces to the sum of the discontinuities at points where the curve intersects the 1-D light-cone. The discontinuity is given by $log(|a^2|/-|a^2|)$ at the limit $a^2 \rightarrow 0$, and equals to $log(-1)$, which can be identified as $\pm i\pi$. The only natural definition is based on same sign of discontinuity so that the integral over a closed curve vanishes and one avoids the introduction of the imaginary unit highly un-natural in hyper-complex context. Note however that there is no obvious objection against complex extension of hyper-complex numbers.

In the case of HQ the pole corresponds to $t^2 - x^2 - y^2 - z^2 = 0$ and it is clear that the only sensible option is the one for which residue integrals over closed curves vanish. This conforms with the physically motivated definition of residue integrals as kind of conformal or hyper-conformal invariants assignable to light-like surfaces X_l^3 having boundaries at light-like 3-surfaces X^7 of $H = M4 \times CP_2$.

2.7 Induction of the (hyper-)octonionic structure

The induction of (hyper-)octonionic structure corresponds to the projection of (hyper-)octonion basis to space-time surface. The normal component of the algebra product could be projected out.

2.7.1 Two manners to induce (hyper-)octonionic structure

The induction of the (hyper-)octonion structure to the space-time surface means that (hyper-)octonionic units $I_k = e_k^A I_A$, where I_A are (hyper-)octonion units multiplied, are projected to the space-time surface

$$I_\alpha = I_k \partial_\alpha h^k \ . \tag{10}$$

If the product of tangent space (hyper-)octonions is defined using the original inner product (no conjugation for $\sqrt{-1}$), the inner product gives induced metric

$$\langle I_\alpha I_\beta \rangle = g_{\alpha\beta} \ , \tag{11}$$

This result is nice but the problem is is that the components of the induced (hyper-)octonion field do *not* generate 4-dimensional (complexified) sub-algebra since the product contains components belonging to the normal space of the space-time surface.

The requirement that the product is automatically tangential to the surface, gives stringent conditions for the space-time surface but is possible to satisfy at least in the case of (hyper-)quaternionic manifolds since the (hyper-)quaternionic sub-spaces of (hyper-)octonions are labelled by CP_2. The assumption that the tangent space of X^4 closes algebraically to (complexified) quaternions makes sense and would assign to each point of resulting 4-surface a point of CP_2.

One can imagine also a second alternative. A four-dimensional algebra property is achieved quite generally if one redefines the (hyper-)octonion product by projecting away the component normal to the space-time surface. This projection operation means that one defines the structure constants of the induced algebra as projections of the structure constants of the octonionic algebra:

$$\begin{aligned} I_\alpha I_\beta &= d_{\alpha\beta}{}^{\gamma} I_\gamma \ , \\ d_{\alpha\beta\gamma} &= d_{klm}\partial_\alpha h^k \partial_\beta h^k \partial_\gamma h^m \ . \end{aligned} \tag{12}$$

One can also induce the algebra to the normal space of the space-time surface and basic formulas are very similar to those encountered in the case of the tangent space induction.

2.7.2 Is the induced (hyper)-octonion structure always associative or co-associative?

The basic motivation behind the entire construction is the idea that either the tangent space of the space-time surface or its normal space could be regarded as an associative algebra. The explicit form of the tangent space associativity conditions

$$I_\alpha(I_\beta I_\gamma) \quad = \quad (I_\alpha I_\beta)I_\gamma \ , \tag{13}$$

reads explicitly as

$$d_{\alpha\beta}{}^{\mu} d_{\mu\gamma}{}^{\delta} \quad = \quad d_{\beta\gamma}{}^{\mu} d_{\alpha\mu}{}^{\delta} \ . \tag{14}$$

In the case of the normal space induction, the conditions are of the similar form. It is convenient to say that space-time surface is co-associative if its normal space possesses associative induced algebra. The situation for the hyper-octonionic induction is essentially the same since only extension by $\sqrt{-1}$ is involved.

The following arguments suggest that associativity/coassociativity indeed holds true. The idea is to use general coordinate invariance to reduce the problem at a given point of the space-time surface to the study of the orthogonal 4+4 decompositions of the standard octonion basis and then explicitly study the induced algebra for various decompositions.

1. Reduction of the problem to the study of the 4+4 orthogonal decompositions of the standard octonion basis

Since a manifestly general coordinate invariant tangent space structure is in question, it seems obvious that it is always possible to find such coordinates that, at a given point of the space-time surface, the components of the octonionic form of H reduce to the standard form having standard multiplication rules of the octonionic generators. This is achieved if at a given point of X^4 one can choose orthonormal coordinates in H such that four coordinate curves are orthogonal to space-time surface and four are parallel to it. The second half of the H-coordinates serves as orthogonal coordinates for the space-time surface. Under these assumptions the algebra of the octonionic components I_k at the point of X^4 is of the standard form and one must only study different 4+4 decompositions of the octonion basis to orthogonal 4-dimensional subspaces to find whether associativity or co-associativity holds true.

In the standard basis, the induction procedure means that one drops away orthogonal components from the product of two octonion units belonging to the tangent space of X^4. Similarly in the case of normal space induction. This means that one can readily look what kind of 4-dimensional algebras are obtained by this procedure and whether they are associative or co-associative.

2. Various 4+4 orthogonal decompositions of the octonionic algebra

There are two cases to be considered according to whether I_0 belongs to the quadruple or not. The crucial observation in what follows is that *any* two imaginary octonion units belong to some of the seven associative triples.

Case A: I_0 belongs to the quadruple

There are two cases to be considered.

i) All three I_k:s belong to same associative triple. In this case, space-time surface has quaternionic structure.

ii) If the third I_k does *not* belong in same triple then all products of I_k lead out from the tangent space. These products vanish in the induced algebra. Thus I_k annihilate each other in the induced algebra and their squares are equal to $-I_0$. The defining relations of the 4-dimensional algebra

$$I_k^2 = -I_0 \ , \quad I_k I_l = 0 \ , \quad k \neq l \ . \tag{15}$$

This is an *associative* algebra representable by 4x4 unit matrix and 3 imaginary matrices with one non-vanishing element i at the diagonal.

There are no other possibilities. These subspaces are *associative* as expected. The result means also that the complements of these spaces are automatically co-associative.

Case B: I_0 does not belong to the quadruple

There are two possibilities also now.

i) There is full associative triple plus one outsider. All products of the outsider with the triple vanish as also vanish the squares of each I_k in the induced algebra structure.

$$I_iI_j = e_{ijk}I_k \ , \quad I_k^2 = 0 \ , \quad I_4I_k = 0 \ , \quad I_4^2 = 0 \ . \tag{16}$$

This algebra is nothing but the algebra generated by the original associative triple endowed with the 3-dimensional cross product and by the fourth element with vanishing square and annihilating the elements of the triple. Since cross product is *non-associative*, also the entire algebra is non-associative.

ii) There is no full associative triple. In this case all products lead out of the system and each algebra generator annihilates itself and others in the induced algebra.

$$I_jI_k \quad = \quad 0 \quad \text{for all } j \text{ and } k \ . \tag{17}$$

This algebra is obviously *associative*. The matrix realization is obtained by taking the four diagonal elements of 4x4 matrix and by replacing them by a nilponent 2x2 matrix.

To conclude, if the assumptions about reducibility of the octonion basis to the standard form are correct, then M^4_+ and CP_2 as a sub-manifolds of $M^4 \times CP_2$ are both associative and co-associative. Same holds also true for the local fiber-base decomposition of $SU(3)$ regarded as a $U(2)$ bundle over CP_2. An example of a non-associative space-time surface is provided by the surface $E^3 \times S^1$, where E^3 is space-like hyperplane of M^4 and S^1 is geodesic circle of CP_2. It seems that non-associative space-time surfaces are not physically interesting in TGD context. One can also consider the induced quaternion structure at 2-dimensional surfaces of a 4-dimensional manifold. The local algebra associated with a given 2-surface is either the algebra of the complex numbers or the algebra generated by two nilpotent elements annihilating each other. For 3-dimensional sub-manifolds one obtains the non-associative algebra defined by the ordinary cross product.

3 (Co-)-Hyper-quaternionicity in $HO \leftrightarrow$ space-time as 4-surface in $M^4 \times CP_2$

This section summarizes the basic vision about number theoretic realization of classical dynamics boiling down to the assumption that space-time surfaces can be regarded as hyper-quaternionic or co-hyper-quaternionic sub-manifolds of hyper-quaternionic $M^8 = HO$. This identification is equivalent with the assumption that space-time surfaces are 4-D surfaces in $M^4 \times CP_2$ so that number-theoretic variant of spontaneous compactification occurs.

3.1 Why hyper-quaternions and -octonions?

Several observations support the view that hyper-quaternions and hyper-octonions are natural from the point of view of TGD.

1. M^4, M^4_+ and its complement in M^4, and the intersection $M^4_+ \cap M^4_-$ of future and past light-cones with different tips can be regarded as 1-dimensional hyper-quaternionic spaces or their projective variants. The new element is that light-cone boundary δM^4_+ takes the role of origin as a projectively invariant set. The existence of light-like numbers basically reflects the failure of number field property of complexified quaternions. If $M^4_+ \cap M^4_-$ is given a metric with Euclidian signature and identifying the points at the boundary with light-like separation are identified, a set having boundary S^2 is obtained. By gluing to it S^2 CP_2 results.

2. The generalized symplectic structure of $H = M^4 \times CP_2$ seems to be intimately involved and generating hyper-quaternion units have representation in terms of covariantly constant antisymmetric forms of M^4 defining "hyper-hyper-Kähler" structure in M^4.

3. $U(2)$ Lie-algebra with central extension term can be identified as HQ. The reason is that $SU(2)$ Lie-algebra commutator differs by a factor of $\sqrt{-1}$ from quaternionic commutator. Thus $U(2)$ has naturally metric with Minkowskian signature.

4. An encouraging hint is the fact that Hamilton-Jacobi coordinates involving two light-like coordinates and complex coordinate and its conjugate are associated naturally with both hyper-quaternionic structure and with the construction of general solutions of field equations [D1].

3.2 How to understand $M^4 \times CP_2$ in the hyper-octonionic context

The hopes of giving $M^4 \times CP_2$ hyper-octonionic structure are meager. This circumstance forces to ask whether four-surfaces X^4 in HO could under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly so that the spontaneous compactification of super string models would correspond in TGD to two different manners to interpret the space-time surface. The following arguments suggest that this is indeed the case.

3.2.1 Hyper-octonions and $SU(3)$

The space of complex structures of the octonion space is parameterized by S^6. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Hyper-quaternions can be identified as $U(2)$ Lie-algebra but it is obvious that hyper-octonions do not allow an identification as $SU(3)$ Lie algebra. Rather, octonions decompose as $1 \oplus 1 \oplus 3 \oplus \overline{3}$ to the irreducible representations of $SU(3)$.

Allowing complexified octonions, a given octonion defines states in 3 and $\overline{3}$. The pairing $3 \otimes \overline{3}$ yields a state in color octet representation and thus an element of $SU(3)$ Lie algebra. This pairing corresponds to the possibility to define an algebra structure combining 3, $\overline{3}$, and $SU(3)$ Lie-algebra to single larger algebra. The algebra product is given by

$$\begin{array}{ll} \left[T^A, T^B\right] = f_C^{AB} T^C \ , & \\ T^A \circ q_i = D^A_{ij} q_j \ , & T^A \overline{q}_i = \overline{D}^A_{ij} \overline{q}_j \ , \\ q_i \circ \overline{q}_j = C_{i\bar{j}A} T^A & q_i \circ q_j = \overline{q}_i \circ \overline{q}_j = 0 \end{array} \tag{18}$$

$C_{i\bar{j}A}$ denote Glebsch-Gordan coefficients.

The "super" Jacobi identities expressing the condition

$$T^A(q_i \circ \overline{q}_j) = (T^A q_i) \circ \overline{q}_j + q_i \circ T^A(\overline{q}_j) \tag{19}$$

are expected to hold true.

3.2.2 CP_2 labels hyper-quaternionic sub-spaces of hyper-octonions for a fixed complex structure

The quaternionic sub-algebras of octonions with fixed complex structure are parameterized by CP_2 just as the complex planes of quaternion space are parameterized by CP_1. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group.

If the preferred octonionic imaginary unit multiplied by $\sqrt{-1}$ must correspond to an anti-symmetric tensor of $H = M^4 \times CP_2$, whose square equals to the metric of H. This tensor is sum of CP_2 part given by CP_2 Kähler form and M^4 part, which is self dual form having in the standard light-cone coordinates $(u = t - z, v = t + z, x, y)$ the representation $J_{uv} = -J_{vu} = 1$, $J_{xy} = -J_{yx} = \sqrt{-1}$. Therefore the ansatz is also consistent with vacuum extremals for which the induced Kähler form vanishes. The selection of the preferred complex structure would be a correlate for the spontaneous symmetry breaking associated with spontaneous compactification in string models.

Assigning to a point of X^4 a quaternionic sub-algebra with a fixed complex structure means assigning to it a point of CP_2. First of all, the choices of a fixed quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex structure (choice of e_1) are labelled by $U(2) \subset SU(3)$. The reason is that the choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which means that one selects the $U(2)$ subgroup of $SU(3)$, or decides what linear superposition of quarks/antiquarks one calls strange quark/antiquark having vanishing isospin. The selection of what quark is regarded as strange quark is invariant under the action of $U(2)$. Hence all possible additions of e_2, e_3 to $1, e_1$ are labelled by $SU(3)/U(2) = CP_2$.

3.3 (Co)-hyper-quaternionic 4-surfaces in HO correspond to space-time surfaces in $M^4 \times CP_2$

The observations about the role of $SU(3)$ and CP_2 imply that HQ and *coHQ* 4-surfaces in HO correspond in one-one manner to 4-surfaces in $M^4 \times CP_2$ and to a general ansatz producing this kind of surfaces.

3.3.1 A map $HO \rightarrow SU(3)$ defining an integrable distribution of hyper-quaternionic planes defines a foliation of $M^4 \times CP_2$ by 4-surfaces

A distribution of HQ (*coHQ*) planes in HO defines its foliation by 4-surfaces X^4 of HO and therefore also that of $M^4 \times CP_2$ if integrability conditions, which state that HQ (*coHQ*) planes define tangent planes of X^4 in the foliation, are satisfied. The M^4 coordinates of X^4 are obtained as the projection of HO to a fixed HQ sub-space of HO whereas the selection of the quaternionic plane defines CP_2 coordinates.

Since *coHQ* case is very similar to HQ case, only HQ case will be discussed explicitly. The hyper-quaternionicity condition states that it is possible to select at each point of HO local $U(2)$ sub-algebra of $SU(3)$. The local algebra is obtained by adjoint action from the standard $U(2)$ algebra at the unit element of $SU(3)$:

$$
\begin{aligned}
T^m_h &= Ad_g(h)(T^m) = g(h)T^m g^{-1}(h) \ , \\
[T^m_h, T^n_h] &= f^{mn}{}_r T^r_h
\end{aligned}
\tag{20}
$$

If the distribution of the hyper-quaternionic tangent planes in HO defined by a map $g : HO \rightarrow SU(3)$ is integrable, a foliation of HO by four-surfaces results, and defines a foliation of $H = M^4 \times CP_2$ so that a 4-parameter family of solutions is obtained. This could perhaps interpreted as stating that the allowed maps $g : HO \rightarrow SU(3)$ are consistent with the bundle structure $\pi : SU(3) \rightarrow CP_2$ in the sense that g induces a bundle structure $HO \rightarrow g^{-1}(CP_2)$. Now however the projection of X^4 to a given fiber need not be a point but can be even four-dimensional surface as in the case of the canonical imbedding of M^4 to H.

3.3.2 A generalization of the solution ansatz to take into account vacuum degeneracy

Vacuum degeneracy is a characteristic feature of Kähler action and implies the presence of infinite number of non-quantum fluctuating zero modes, which do not contribute to the metric of the configuration space. Also the solution ansatz should have analogous degeneracy. This suggests that the solution ansatz assuming that the preferred imaginary unit is same in entire HO is too restricted and should be made local. This means local $S^6 = G_2/SU(3)$ labelling different orientations of the imaginary unit.

Thus the degeneracy due to the possible local choices of the hyper-octonionic complexification corresponding physically to the choice of the plane of non-physical polarizations, becomes a candidate for this degeneracy and would expand the group of local symmetries from $SU(3)$ constrained by the integrability conditions to the entire automorphism group G_2 of octonions. The local fixing of complexification in HO means the fixing of a map $f : OH \rightarrow S^6$. Probably this map could satisfy some constraints forced by the absolute minimization. If the choice of f is completely free, the integrability conditions would be invariant under $G_2 \subset SO(7)$ automorphisms. The maps $f : HO \rightarrow S^6$ and $g : HO \rightarrow SU(3)$ could be interpreted as a map $h : HO \rightarrow G_2$ in the local trivialization $G_2 = S^6 \times SU(3)$.

3.3.3 Also $coHQ$ 4-surfaces are needed

It seems that also the dual solutions for which the normal space is hyper-quaternionic must be allowed since otherwise it is not possible to understand CP_2 type extremals, which are definitely quaternionic objects. The four parameters labelling the solutions become space-time coordinates for the dual solution whereas the space-time coordinates for the solution parameterize dual solutions.

The surfaces at which the induced metric becomes light-like might allow to glue together solutions corresponding to different functions g and f. The intuitive expectation is that the light-likeness for 3-D surfaces should correspond to the number-theoretic light-likeness of a hyper-quaternionic space-time coordinate. If the hyper-quaternionic functions are rational functions, 3-D light-like causal determinants can appear as a generalization of the poles of a rational function.

3.3.4 Why not octonion analyticity instead of hyper-octonion analyticity?

Mind must be kept open also for the octonionic variant of the solution ansatz might make sense. $HO = M^8$ can be replaced with $O = E^8$, space-time surfaces as hyper-quaternionic sub-manifolds can be replaced with quaternionic sub-manifolds, and the map $M^8 \rightarrow M^4 \times CP_2$ can be replaced with a map $E^8 \rightarrow M^4 \times CP_2$.

The map $O \rightarrow M^4$ is defined as the canonical projection to Q followed by the multiplication of quaternionic imaginary units with $\sqrt{-1}$. Hence the possibility that octonionic ansatz might make sense must be left open.

The differences between the two solution ansätze become obvious when the hypothesis that infinite hyper-octonionic primes are representable in terms of hyper-octonionic polynomials is discussed in the chapter [E3]. As found, these notions of primeness differ in a profound manner, and the fact that hyper-octonionic primes allow an interpretation as Minkowskian 8-momenta encourages to think that they define the correct option.

3.4 Integrability conditions

If the distribution of hyper-quaternionic planes are identifiable as tangent planes of space-time surface $X^4 \subset HO$, commuting tangent vector fields $\partial\alpha$ associated with X^4 coordinate variables x^α exist. Integrability conditions express the commutativity of these vector fields lifted to HO vector fields. Note a that $X^4 \subset H = M^4 \times CP_2$ imbedding exists by definition, and it is only the integrability to a 4-surface in HO, which requires additional conditions, hoped to be equivalent with field equations, to be satisfied.

3.4.1 Induction of $SU(3)$ Lie algebra vector fields to HO and tangent plane

$SU(3)$ Lie-algebra generators T^A define vector fields in $SU(3)$. The dual forms ω_A can be induced to HO and either $HO = M^8$ Minkowski metric m_{ij} can be used to lift them to vector fields of HO by the index raising operation

$$\hat{T}^{Ai}\partial_i \quad = \quad m^{ij}\omega_{Ak}\partial_j g^k \partial_i \ . \tag{21}$$

The forms ω_A induced to HO can in turn be induced to forms in the local hyper-quaternionic tangent plane and the metric of the tangent plane allow to transform these forms to vector fields in X^4. The natural tangent plane metric is the metric $g_{\alpha\beta}$ induced from the metric of HO. This metric could in turn be induced from that of $H = M4 \times CP_2$. Also the the Kähler form of CP_2 could be induced to HO and could perhaps serve as representation for e_1.

The integrability conditions in HO should be equivalent with the field equations defined by Kähler action and in these equations the induced metric and Kähler form of H appear.

3.4.2 The analogy of integrability conditions with those for a flat connection

Integrability implies that X^4 has tangent vector field basis ∂_α in HO. It is possible to express tangent vector fields $\partial_\alpha h^k$ as linear combinations of HO vector fields $\hat{T}^m$ defined the local $U(2)$ Lie-algebra generators T_h^m, where the subscript h tells that the $U(2)$ subalgebra depends on HO coordinate h and is obtained by the adjoint action:

$$\partial_\alpha = A_{\alpha m}\hat{T}_h^m \ .$$

The interpretation of A as an analog of $U(2)$ gauge potentials suggests itself. The difference is that $\hat{T}^m$ does not represent $SU(3)$ vector field but induced HO vector field.

The integrability conditions express the commutativity condition $[\partial_\alpha, \partial_\beta] = 0$. If $\hat{T}_h^m$ would represent $SU(3)$ vector field, the integrability conditions would translate to the flatness of $U(2)$ connection:

$$dF \quad = \quad dA + [A, A] = 0 \ . \tag{22}$$

In the recent case the conditions have a more complex but analogous form:

$$\left[A_{\alpha m}(\hat{T}_h^m \circ A_{\beta n}) - A_{\beta m}(\hat{T}_h^m \circ A_{\alpha n})\right]\hat{T}_h^n + A_{\alpha m}A_{\beta n}\left[\hat{T}_h^m, \hat{T}_h^m\right] = 0 \ . \tag{23}$$

The generators $\hat{T}^m$ defining $U(2)$ basis differ by a local $SU(3)$ gauge transformation and by the effects caused by induction from the standard basis. An analog of with flat $U(2)$ connection is obvious. This brings in mind the structure of CP_2 as coset space: also in this case $U(2)$ acts as a local gauge group and permutes points inside $U(2)$ cosets. Now these cosets wold be replaced by cosets of local $SU(3)$.

To gain a better understanding of what is involved it is good to clarify what are the basic bundle structures involved.

1. $SU(3) \to CP_2$ defines a $U(2)$ bundle and is essential for H picture.

2. The tangent bundle $T(G) \to G = SU(3)$ and corresponding cotangent bundle $T^*(G)$ are essential for HO picture and appear in the integrability conditions. $T(G) \to G = SU(3)$ bundle structure is induced to give bundle with base space HO by mapping it first to cotangent bundle by assigning to vector fields their duals, inducing the cotangent bundle by standard procedure, and lifting it back to (possibly sub-) vector bundle of the tangent bundle of HO. The induction procedure for vector fields is what brings in dynamics involving the metric induced from H.

3. $U(2)$ is identified as a sub-manifold of the base G at given point of CP_2, and $U(2)$ tangent space vector fields are induced to vector fields in local hyper-quaternionic spaces and integrability conditions imply that these vector fields define tangent space basis in X^4.

3.5 How to solve the integrability conditions?

In the following some attempts to understand integrability conditions are made. After more or less ad hoc attempts an ansatz based on hyper-octonion analyticity is proposed.

3.5.1 Guesses for the solution of integrability conditions

A trivial vacuum solution with constant CP_2 coordinates results if the local trivialization $SU(3) = U(2)\times CP_2$ is induced to HO by the map g and space-time surfaces correspond to inverse images of $U(2)$. Hyper-quaternionic sub-space is same at each point of X^4 now. Any invertible map g defines trivial vacuum solutions in this manner. Obviously, non-trivial solutions cannot be consistent with the local foliation $SU(3) = U(2) \times CP_2$.

One might wishfully think that the expression for a flat connection generalizes and defines a solution of the integrability conditions also now. This would boil down to the replacement

$$A_\alpha = h^{-1}\partial_\alpha h = (h^{-1}\partial_\alpha h)_m T^m h \to \quad (h^{-1}\partial_\alpha h)_m \hat{T}^m \ . \tag{24}$$

h should be $U(2)$ valued map $X^4 \to U(2)$ for each sheaf of the foliation and would define $U(2)$ coordinate for X^4.

A second possibility popping in mind is that the induced vector fields in X^4 define the original $U(2)$ Lie-algebra for the solutions of the integrability conditions apart from the scaling of Lie-algebra generators, and thus of structure constants, by functions of $U(2)$ invariants. This would mean that the effect of the adjoint action and index raising operation with HO metric preserves $U(2)$. In this case A would define a genuine $U(2)$ connection and integrability conditions would state its flatness.

3.5.2 Do hyper-octonion analytic maps $HO \to HO$ define solutions to the integrability conditions?

HO picture raises two challenges. First of all, a general solution to the integrability conditions should be found. Second task is to demonstrate that field equations determined by Kähler action are under some additional conditions equivalent with the solution.

1. The general hyper-octonion analytic ansatz

Hyper-octonion analytic maps with real coefficients from OH to itself suggests themselves as candidates for this kind of maps. The key observation is that it is possible to assign to a map $HO \rightarrow HO$ a map $HO \rightarrow SU(3)$. HO tangent space has $1 \oplus 1 \oplus 3 \oplus \overline{3}$ decomposition so that the tensor product of $3 \otimes \overline{3}$ gives a color octet vector field identifiable as an element of local $SU(3)$ Lie algebra. The exponentiation of this vector field defines an element of local $SU(3)$ defining in HO a distribution of hyper-quaternionic tangent planes.

If hyper-octonion analyticity guarantees integrability conditions, a foliation of HO by 4-surfaces X^4 and hence of $H = M^4 \times CP_2$ results. There is a definite analogy with spontaneous compactification in that TGD in flat 8-D non-compact space HO would be equivalent TGD in $M^4 \times CP_2$.

What might be called simple hyper-quaternion analytic maps with real Laurent coefficients, are of form

$$h_0 + \overline{h} \rightarrow a(h)h_0 + b(h)\overline{h}$$

as is easy to find by looking what happens in the map $h \rightarrow h^2$ and by generalizing using induction.

Also more general maps defined as composites of a local G_2 automorphisms $\hat{f}(h)$ performed for the imaginary part of h and followed by a a simple hyper-octonion analytic map are possible and give rise to the result

$$h_0 + \overline{h} \rightarrow h_0 + \hat{f}_h(\overline{h}) \rightarrow a(h_0, |h|^2)h_0 + b(h_0, |h|^2)\hat{f}_h(\overline{h}) \ . \tag{25}$$

This means that the simple hyper-octonionic map $\hat{g}$ is is replaced with the composite $\hat{g} \circ \hat{f}$. $\hat{f}$ is HO-local $G_2 \subset SO(7)$ rotation, and defines an element of $f(h)$ of S^6 assuming that the imaginary part is thought of as being obtained from some fixed imaginary unit by G_2 element.

Obviously, the HO analytic maps involving local G_2 automorphism represent a generalization of anti-analytic maps. In HQ analytic case the corresponding degree of freedom corresponds to a local $SO(3)$ rotation.

In the case that the $\hat{f}_h$ depends only on the direction of $\overline{h}$,

$$\hat{f}_h = \hat{f}(\frac{\overline{h}}{|\overline{h}|}) \ , \tag{26}$$

$\hat{f}$ commutes with $\hat{g}$:

$$\hat{g} \circ \hat{f} = \hat{f} \circ \hat{g} \ . \tag{27}$$

Under this condition the map $HO \rightarrow S^6$ reduces to a map $S^6 \rightarrow S^6$. Physically the commutativity conditions is highly attractive as a symmetry giving hopes about consistency with field equations.

The solution ansatz defines an element of local automorphism group G_2 in the local trivialization $G_2 = S^6 \times SU(3)$. The imaginary part of the hyper-octonion forms a 7-D representation of G_2 and $7 \otimes 7$ tensor product defines an element of the Lie-algebra of G_2 and hence the tensor product defines a map to local G_2. The obvious question is whether the two elements of G_2 defined in this manner are identical.

The solution ansatz involves two $U(2)$ algebras. The first one corresponds to the hyper-quaternionic tangent space and for this representation hyper charge generators is represented by unit matrix. This algebra is very much analogous to electro-weak $U(2)$. Second representation of $U(2)$ algebra results from $2 \times \overline{2}$ tensor product. It would not be too surprising if these algebras could be mapped to each other in the sense that octonionic products for $2+\overline{2}$ would give the hyper-quaternionic $U(2)$.

2. Is the reduction to Lie-algebra level possible?

The $SU(3)$ generator given by the tensor product $3\times\overline{3}$ is of form

$$\begin{aligned} X &= b^2(h)\hat{h}^i_3\hat{h}^j_{\overline{3}}C_{ijA}T^A \ , \\ \hat{h} &= \hat{f}_h(h) \ . \end{aligned} \tag{28}$$

The Lie-algebra element at a given point of HO differs only by the scaling factor $b^2(h)$ for different maps when the choice of imaginary unit is kept fixed. Therefore, at a given point of HO the values of $g(h)$ for various hyper-quaternion analytic maps belong to the same one-parameter sub-group $U(1)_h$ determined by $X(h)$.

This reduction and the fact that g is otherwise arbitrary and can be arbitrary near to identity map raises the hope that it is enough to consider the conditions infinitesimally so that $Ad(g)-1$ reduces infinitesimally to a commutator in Lie algebra. If this is the case, the conditions are satisfied if X is annihilated by the adjoint action of the $U(2)$ generators T^m and would thus define an $U(2)$ invariant vector field in X^4. Taking into account the universal nature of X this not be surprising. Since b^2 is $U(2)$ invariant function so that the remaining universal vector field should be invariant under local $U(2)$ and analogous to the $SU(3)$ invariant vector fields in CP_2.

3. Does hyper-quaternion analyticity make sense?

A wishful thinker, inspired by the idea about strings $\rightarrow$ TGD transition as $H_2--R\rightarrow HO--HQ$ transition, might hope that also hyper-quaternion analyticity makes sense. HO allows the decomposition $HO=HQ+\sqrt{-1}e_3Q$ with coordinate $h+\sqrt{-1}e_3q$. Assume that X^4 allows HQ manifold structure and let x be HQ coordinate for X^4 (this is quite strong an assumption). Continuing in wishful spirit, $h=f(x)$ could be HQ analytic function whereas q could be obtained from HQ analytic function $g(x)$ by multiplying its real part with $\sqrt{-1}$. HQ analyticity would mean that the function would be of general form $x_0+\sqrt{-1}\overline{x}\rightarrow a(x_0,|\overline{x}|^2)x_0+b(x_0,|\overline{x}|^2)\sqrt{-1}O(x)\overline{x}$. Here $O(x)$ is a local $SO(3)$ rotation.

3.6 $HO-H$ duality and the variational principle behind HO dynamics?

$HO-H$ duality suggests that HO dynamics is derivable from a variational principle. There are some arguments suggesting that $HO-H$ duality could be regarded as a kind of color–electro-weak duality. This duality is also supported by the basic facts about 8-dimensional vector and spinor representations of $SO(7,1)$. A reasonable candidate for the variational principle is as a Dirac action for 8-D hyper-octonionic spinor fields in 8-D hyper-octonion space.

3.6.1 $HO-H$ duality as color-electro-weak duality?

One can wonder whether the imbedding defines naturally classical electro-weak and color gauge potentials at the space-time surface. One can also wonder how the two dual pictures corresponding to HO and $M^4\times CP_2$ relate to this.

1. The projections of the duals of $SU(3)$ Lie algebra generators lifted to vector fields at the space-time surface would be natural candidates for classical color gauge fields. If the $U(2)$ algebra is preserved in the induction procedure, the integrability conditions imply the vanishing of a genuine $U(2)$ gauge field. A natural interpretation would be as an electro-weak gauge field. Electro-weak gauge fields would not appear in HO picture.

2. In H picture electro-weak gauge potentials can be induced from the spinor connection of $M^4 \times CP_2$. The projections of Killing vector fields of $SU(3)$ in CP_2 define analogs of gluons but since they do not appear in the modified Dirac equation for induced spinors nor in the Dirac equation for imbedding space, one might argue that genuine gluon fields are not in question.

These observations give some hints about the concrete physical interpretation of $HO - H$ duality. For HO representation of the space-time surface classical color gauge fields are naturally present whereas for H representation this is the case only for electro-weak gauge fields. A vague hunch about this kind of duality has been present in TGD framework from beginning. For instance, induced spinor fields do not carry color as a spin like quantum number whereas color triplet and antitriplet occur naturally in HO representation and could multiply the solutions of the modified Dirac equation in HO.

If this duality makes sense, H picture could correspond to the description of hadron physics using hadrons as basic particles and using the current algebra defined by the electro-weak currents. HO picture would correspond to QCD approach based on the use of color currents. Color confinement might be seen as an impossibility to detect color in the experiments based on $M^4 \times CP_2$ description.

There is an obvious objection against this picture. HO spinor fields lack completely spin and ew spin indices. On the other hand, H spinors lack color as spin index. In H picture, classical color charges are however well-defined and color emerges at configuration space level as a kind of orbital degree of freedom. This leads to the idea that in HO description using the dual CHO of configuration space CH, spin and ew spin correspond to configuration space orbital degrees of freedom and color to spin like quantum numbers. $HO - H$ duality would permute orbital and spin degrees of freedom for configuration space spinor fields. This idea, to be be developed in more detail later, leads to quite nice understanding of dualities of hadron physics.

One can ask how the $H - HO$ duality relates to the duality between the proposed dual descriptions using partonic 2-surfaces and interiors of corresponding 3-surfaces (7-3 duality). Super-canonical conformal weights expressible in terms of zeros of zeta are associated with partonic 2-surfaces and zeros of ζ seem to be dual to primes much like momentum is dual to position.

3.6.2 The variational principle behind HO dynamics?

A genuine dynamics in HO degrees of freedom suggested by $HO - H$ duality could help in fixing the details of the scenario. There are several hints as regards to the nature of the HO dynamics.

1. The variational principle for the maps $HO \rightarrow HO$ dynamics should be something very simple since hyper-octonion analytic maps with a local G_2 invariance should provide solutions of field equations.

2. $1 + 1 + 3 + \overline{3}$ decomposition of octonions suggest that the variational principle could be something which might be regarded massless Dirac action for a spinor field of HO as hyper-octonionic argument and itself having interpretation as octonion. The interpretation of the spinor components would be as analogs of leptons and quarks which do not however carry spin and electro-weak spin but only color. Color octet field, analogous to a gluon field, could be constructed as a bi-linear of $3 + \overline{3}$ part of this field. This picture would conform with color-electro-weak duality.

1. Spinors as octonions

The idea is that (hyper-)octonion analytic function can in some sense be regarded as 8-component spinor field with the units of octonion representing components of the spinor field.

The problem is that octonionic gamma matrices are necessary 2×2-dimensional in octonionic sense so that 2-D octonionic spinors would be required.

The difficulty can be circumvented by the replacement of the massless Dirac equation by the equation obtained by multiplying it with γ_0. This means that gamma matrices are replaced with α matrices $\alpha_0 = 1$ and $\alpha_i = \gamma_0\gamma_i$. In fact, Dirac treated Dirac equation just in this manner and interpreted α matrices as components of four-velocity operator. The space-like alpha matrices satisfy the anti-commutation relations

$$\{\alpha_i, \alpha_j\} = -2m_{ij} \ . \tag{29}$$

The breaking of the manifest Lorentz invariance is natural in the number theoretic context where $SO(1,7)$ reduces to G_2.

The motivation for this trick is that α matrices allow a representation as quaternionic/octonioni units: $\alpha_k = e_k$. In the case of octonions the representation is of course not matrix representation so that spinors in the usual sense cannot be defined. The idea is that the components of spinor basis at a fixed point are identifiable as alpha matrices so that spinor fields are identifiable as octonion valued functions of octonion variable. The interpretation as a super-symmetry is obviously possible.

The counterpart of the massless Dirac equation reads in the octonionic case as

$$(\partial_t + \alpha^i\partial_i)\Psi = 0 \ . \tag{30}$$

The solution is obtained by generalizing the usual solution ansatz

$$\Psi = (\partial_t - \alpha^i\partial_i)\Psi_0 \ , \tag{31}$$

where Ψ_0 is octonion analytic function with real Laurent coefficients interpreted as spinor. Dirac equation reduces to Laplace equation for Ψ_0:

$$(\partial_t^2 + \nabla^2)\Psi_0 = 0 \ . \tag{32}$$

This equation is satisfied separately by each component of Ψ_0 as direct check of the Laplace equation for the few lowest powers of the octonion variable demonstrates. The result, which is a direct generalization of the corresponding result of complex analysis, must reduce to the Cauchy Riemann equations satisfied by real-analytic octonion power series. Hence the vague idea about octonion analyticity as a counterpart of ordinary analyticity is beautifully realized.

In the case of hyper-quaternions and -octonions the introduction of $\sqrt{-1}$ is necessary but only changes Laplace equation to massless d'Alembert equation. In both cases the octonion conformal flatness of the space meaning that metric is apart from a conformal factor proportional to the flat space matric is absolutely essential for the ansatz to work since otherwise the Hermitian metric does not disappear from the Laplace equation

$$\partial_\alpha \left[g^{1/8} g^{\alpha\beta} \partial_\beta \right] \Psi_0 = 0 \ .$$

Note that octonionic spinor must be defined as 1/4-density in order to achieve the elimination of the metric.

This Dirac equation can be deduced from a variational principle. The point is that γ^0 appearing in the definition of $\bar{\Psi}$ multiplies gamma matrices to give α matrices and thus disappears from the action.

2. Generalization of the solution ansatz to 2-component octonionic spinors

The solution ansatz generalizes also to the octonionic representation of gamma matrices. The minimal octonionic representation of gamma matrices requires two-component octonionic spinors with the minimal octonionic representation of gamma matrices being be defined as

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \gamma_i = \begin{pmatrix} 0 & e_i \\ -e_i & 0 \end{pmatrix} \ , \ i = 1, ..., 7 \ . \tag{33}$$

In this case the solution ansatz is of form $\Psi = \gamma^k \partial_k \Psi_0$, where the components of Ψ_0 are real-analytic functions of octonion argument. The solution involves two real-analytic functions of octonion variable. It seems however possible to pose a generalized Weyl condition

$$P\Psi_0 = 0 \ , \quad P = \frac{1}{2}\begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} \ , \quad \epsilon = \pm 1 \ . \tag{34}$$

Not surprisingly, 2-dimensional Weyl condition allows to eliminate the second spinor chirality and it is therefore possible to formulate hyper-octonion analyticity using ordinary Dirac equation. Conserved quark and lepton chiralities for H spinors are counterparts for the two chiralities now.

The interpretation of (hyper-)quaternionicity condition in this framework is that induced gamma matrices generate (complexified) quaternionic algebra and thus allow a matrix representation obtained by representing quaternion units by complex 2×2 matrices.

3. How to guarantee local G_2 invariance?

The dynamics should allow local G_2 invariance possibly restricted by the commutativity condition. The local G_2 rotation of the imaginary units does not commute with the derivatives and it seems that the introduction of G_2 gauge potential acting on HO spinors cannot be avoided. The introduction of a genuine gauge field does not look an attractive idea. Fortunately, the gauge potential could be pure gauge and obtained by exponentiating the G_2 Lie-algebra valued field obtained as a bilinear of $7 \otimes 7$ tensor product of imaginary parts of octonionic spinor fields. Thus there would be nice internal consistency.

4. $SO(1,7)$ structure contra G_2 structure

One might ask whether the spinors, conjugate spinors, and their octonionic argument could correspond to the real spinor representation 8_s, its conjugate $\overline{8}_s$, and vector representation 8_v of $SO(1,7)$ so that the tensor product of imaginary parts of spinor and its conjugate would give the G_2 group element defining the gauge potential.

This interpretation requires a matrix representation of Clifford algebra to realize $SO(1,7)$. For octonionic representations $SO(1,7)$ does not however act as the group leaving the octonion norm invariant. In particular, the octonionic counterparts of sigma matrices are not matrices anymore and do not generate $SO(1,7)$ algebra. $SO(1,7)$ is replaced by G_2 and color group appears naturally too in accordance with the notion of color–electro-weak duality.

The octonionic counterpart of $SO(1,7)$ Lie-algebra makes however sense [39]. The infinitesimal transformations leaving the octonion norm invariant correspond to infinitesimal, purely imaginary octonionic scalings $o \rightarrow (1 + \epsilon a)o$, where a is a purely imaginary octonion. This is easy to see using the multiplicativity of the octonion norm. By the non-associativity of octonions, Jacobi identities for this 7-dimensional algebra are not satisfied so that it is not a Lie-algebra but what is known as Malcev Algebra. The local gauge pseudo-group would correspond to local purely imaginary octonionic scalings. Unfortunately, these transformations lead out from hyper-octonionic sub-space so that they do not seem to have application to the recent case.

5. Induced HO spinors at space-time surface

Whether space-time surfaces understood as surfaces of H might allow quaternionic or hyper-quaternionic structure or perhaps an Abelian variant of this kind of structure obtained by assuming that hyper-quaternion and -octonion units are commutative, and whether also modified Dirac equation could allow solutions expressible as generalized analytic functions, has been a subject of long lasting optimistic speculations. The conceptual difficulties have been related to the lack of the understanding of the Hermiticity concept and to the problems caused by the non-commutativity. If $HO--H$ duality makes sense these speculations find a natural context in HO picture.

If the metric of X^4 induced from HO is hyper-quaternionic, which means that it is hyper-conformally equivalent with flat Minkowski *resp.* Euclidian metric for 4-surfaces *resp.* their duals, massless Dirac equation for the induced HO spinors identifiable as (hyper-)quaternions in the local (hyper-)quaternionic sub-space makes sense. Induced gamma matrices can be regarded as projections of the HO octonion basis. The solutions of the Dirac equation are given by hyper-quaternion analytic power series with real coefficients. These spinors represent hyper-quaternion analytic maps of space-time surface, and thus leave the interior of space-time intact but deform its boundaries and thus act as dynamical symmetries and induce corresponding symmetries at the level of H.

3.7 How extremization of Kähler action could correspond to the hyper-quaternionicity of 4-surface?

It is natural to close the discussion with an attempt to understand how the proposed speculative ideas might be realized at quantitative level. The basic questions are following.

1. Is hyper-quaternionicity and its co-property consistent with the extremization of Kähler action in the proposed sense?

2. Is hyper-octonionic WZW action plus Dirac action for generalized ribbon diagrams dual to the Kähler action plus modified Dirac action?

There are some general arguments supporting the idea that hyper-quaternionicity *resp.* its co-property are consistent with the extremization of Kähler action *resp.* its dual.

3.7.1 Extrema minimize algebraic complexity

Complexity is minimized for the extremals of any action, and this reduction of complexity could have an algebraic counterpart. Quite generally, at the extrema of any action the tangent space of the space of all possible configurations reduces effectively to a point since the first variation of the action vanishes. The reduction of the complexified octonionic algebra generated by the tangent space-vectors of the generic space-time surface to a complexified quaternionic sub-algebra could be an algebraic correlate for this dimensional reduction.

3.7.2 Minimization of Kähler action as minimimization of non-commutativity

Non-commutativity represents complexity and should be minimized for the extrema of the Kähler action. A measure for the non-commutativity of the tangent space algebra can be defined as follows. Since the preferred local hyper-octonionic unit e_1 is fixed for the allowed variations, the measure must be constructed from the projections of the tangent space vectors to $3+\overline{3}$ sub-space of the complexified octonionic algebra associated with the hyper-octonionic spinor field.

The commutator for the projections of tangent vector fields to $3+\overline{3}$ part of algebra is anti-symmetric in tangent space indices and defines a generator of color Lie-algebra as

$$[e^{(3)}_\alpha, e^{(\overline{3})}_\beta] \rightarrow f^A_{\alpha\beta} T_A \quad . \tag{35}$$

A natural net measure for non-commutativity would be the quantity

$$L = f^A_{\alpha\beta} f^{\alpha\beta}_A \tag{36}$$

integrated over the space-time surface: something very much akin to YM action associated with classical color gauge fields. When the tangent space algebra reduces to a complexified quaternion algebra, the non-commutativity is minimized since the contribution from the normal space vanishes.

On the other hand, Kähler action can be identified as color YM action for induced color gauge potentials defined by the projections of color Killing vector fields. This follows from the fact that classical color gauge field corresponds to the product $J_{\alpha\beta} H^A$ of Kähler form with the Hamiltonians of the color algebra, and from the identity $H^A H_A = 1$. Thus Kähler action could be interpreted as a measure for the non-commutativity of the tangent space algebra and would be minimized in the proposed sense for the hyper-quaternionic 4-surfaces.

The same minimization of complexity occurs also for co-hyper-quaternionic surfaces for which the commutators of the tangent space vectors belong to the complexified normal space. Hence one expects that the extremization of Kähler action works quite generally.

3.7.3 Minimimization of non-associativity

(Co-)-hyper-quaternionic sub-manifolds are (co)-associative. This means that the associator of any three tangent space vector fields vanishes:

$$A_{\alpha\beta\gamma} = e_\alpha(e_\beta e_\gamma) - (e_\alpha e_\beta)e_\gamma = 0 \quad . \tag{37}$$

The complete antisymmetry of the associator with respect to its indices suggests that associativity due to hyper-quaternionicity is equivalent with the existence of an identically vanishing topological current

$$j^\alpha = k\epsilon^{\alpha\beta\gamma\delta} A_{\beta\gamma\delta} \quad . \tag{38}$$

The topological current

$$j^\alpha = \epsilon^{\alpha\beta\gamma\delta} J_{\beta\gamma} A_\delta \tag{39}$$

defined by the Chern-Simons term associated with induced Kähler form is the only candidate for this current, and indeed plays a key role in the construction of the known extrema of field equations [D1]. This current vanishes when the dimension D of the CP_2 projection of the space-time surface satisfies $D \leq 2$. For $D \leq 2$ associativity would hold true and $D > 2$ would correspond to co-associativity. Recall that the dimension of CP_2 projection plays serves as a classifier for the known extremals of Kähler action [D1].

4 Is the number theoretic dynamics consistent with the absolute minimization of Kähler action?

In string model hyper-analytic/analytic dynamics is consistent with the minimal surface property. In TGD framework the question is whether the dynamics defined by the Kähler action is equivalent with the dynamics of the hyper-quaternionic ansatz or some other variant of the number theoretical dynamics.

4.1 The problem

The number theoretic ideas force to ask whether the absolute minimization of Kähler action is really the fundamental variational principle and whether absolute minimization is necessary at all as a variational principle. If it is, as the wishful thinker wants to believe, the challenge is to prove the fundamental role of Kähler action, and to understand how absolute minimization relates to hyper-quaternionic or co-hyper-quaternionic structure of the space-time surface.

The idea about the reduction of field equations for the absolute minima to essentially algebraic statements in HO is not completely outlandish idea: the known large variety of solutions of field equations are indeed satisfied for purely algebraic reasons [D1].

Of course, one must ask how generally the equivalence of number theoretic and Kähler dynamics is.

1. Could it be that extremals, or absolute minima of Kähler action, or perhaps only maxima of Kähler function, correspond to hyper-octonion analytic maps of HO? Or do hyper-octonion analytic solutions correspond to asymptotic patterns characterized by the vanishing of Lorentz Kähler 4-force, whereas more general solutions satisfying the integrability conditions would represent non-asymptotic dynamics. Be as it may, the minimum requirement is that hyper-analyticity is consistent with the conservation laws associated with Kähler action.

2. One can also ask whether the 3-dimensionality of the 3-space has a number theoretic interpretation. Light-like causal determinants X^3_l have vanishing metric determinant and this property is invariant under HQ analytic transformations which induce local scaling of the metric determinant. Could it be that only light-like 3-surfaces allow HQ analytic transformations as symmetries and that these symmetries are equivalent with Kac-Moody super-conformal symmetries? Could it be that these symmetries corresponds to HQ analytic transformations of X^4 acting as mere coordinate transformations in the interior?

What can one then do?

1. The first things to come in mind is the attempt to show that Kähler action is invariant under transformations of foliation of $M^4 \times CP_2$ by 4-surfaces induced by hyper-octonion analytic maps of HO or transforms by a kind of Weyl scaling factor.

2. One could also wonder whether the absolute minimization of Kähler action corresponds to some variational principle in HO. This action principle should have sub-group of local G_2 as dynamical symmetries.

4.2 Does Kähler action allow a generalized conformal invariance?

Generalized conformal invariance in dimension $d = 4$ is a generalization of conformal invariance in dimension $d = 2$, where the metric tensor transforms by a multiplicative factor in conformal transformations. Note that also the ordinary Maxwell action possesses conformal invariance. This conformal invariance could be realized in terms of hyper-octonion analytic maps of HO possibly inducing hyper-quaternion analytic maps of space-time surfaces in the foliation and thus corresponding maps of $H = M^4 \times CP_2$.

In algebraic context the norm for the vectors in n-dimensional algebraic extension is obtained as a special case of n-linear function of n-vectors rather 2-linear function of 2 vectors. This means that in dimension $d = 4$ the algebraically natural quantity is the four-tensor

$$M_{\alpha\beta\mu\nu} = g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu} \ , \tag{40}$$

which represents the metric in the space of the induced sigma matrices.

The conformal covariance of M is however un-necessarily strong condition since it is the contraction of M with the quantity $K_{\alpha\beta\mu\nu} = J_{\alpha\beta}J_{\mu\nu}$ having vanishing conformal weight, which defines Kähler action density. One can express square root of metric determinant as square root of the quantity $\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha_1\beta_1\mu_1\nu_1}M_{\alpha\alpha_1\beta\beta_1}M_{\gamma\gamma_1\delta\delta_1}$ and the metric determinant disappears from field equations. Kähler action density can be written in terms of these quantities as

$$\begin{aligned} L_K &= M^{\alpha\beta\mu\nu}K_{\alpha\beta\mu\nu}\sqrt{det(g)} \equiv M \cdot K\sqrt{g} \ , \\ det(g) &= \epsilon^{\alpha\beta\gamma\delta}\epsilon^{\alpha_1\beta_1\mu_1\nu_1}M_{\alpha\alpha_1\beta\beta_1} \ . \end{aligned} \tag{41}$$

Both metric and $M_{\alpha\beta\mu\nu}$ might fail to transform multiplicatively. What is needed is that $M \cdot K$ and $\sqrt{g}$ transform with opposite conformal factors in algebraically analytic deformation. Unless the deformation reduces to a canonical transformation, it modifies induced Kähler form and $M_{\alpha\beta\mu\nu}$ cannot transform multiplicatively if the deformation is generalized conformal transformation. In dimension $d = 4n$ the counterpart of the tensor M is the tensor defined by the inner products for the antisymmetrized products of $2n$ gamma matrices whereas K corresponds to a tensor power of Kähler form.

One can eliminate all un-necessary complications due to the signature of the metric and possible existence problems (vielbein involves square roots of the metric components and need not exist except in special points) by reducing the study of the generalized conformal symmetries to the study of the behavior of quantities $M \cdot K$ and $\sqrt{g}$ under algebraically analytic transformations.

4.3 Generalized conformal invariance and Euler-Lagrange equations

The realization of the generalized conformal invariance at the level of the field equations boils down to the vanishing of certain tensor contractions. For minimal surfaces encountered in string models algebraic analyticity means that second fundamental form is holomorphic tensor having only components of type (2,0) and (0,2) whereas metric is tensor of type (1,1). Therefore the field equations, which state the vanishing of the contraction of metric and second fundamental form, are satisfied. In present case the situation is very much like this.

1. For the families of extremals studied in [D1] Lorentz Kähler 4-force vanishes. The interpretation is as a space-time correlate for the absence of dissipation for asymptotic self-organization patterns. Absolute minimization of Kähler action would correspond to the second law of thermodynamics. One the known solutions represent only asymptotic patterns, which by the effective 2-dimensionality might be however quite enough for the needs of quantum TGD.

2. Field equations involve the contraction $T^{\alpha\beta}D_\beta\partial_\alpha h^k$ of the energy momentum tensor $T^{\alpha\beta}$ with the second fundamental form defined by the covariant derivatives of the coordinate gradients $\partial_\alpha h^k$. This contraction vanishes for all known solutions of field equations. Field equations contain also terms involving the contraction of the Kähler gauge current j_K^α with the gradients of CP_2 coordinates. The contraction vanishes for all known solutions of field equations because Kähler current either vanishes or is light like.

3. Field equations are satisfied if energy momentum tensor has only non-diagonal components. If the induced metric contains no diagonal components, energy momentum tensor is indeed of the required type. The non-diagonality of the induced metric however implies minimal surface

property, and there must exist and indeed exist ("massless extremals") more general solutions with non-diagonal energy momentum tensor but with metric having diagonal components. This solution family is invariant under canonical transformations of CP_2 and the solutions have 2-dimensional CP_2 projection. Field equations are satisfied because energy momentum tensor is a light like tensor: the contraction of light like diagonal component of energy momentum tensor with the second fundamental form vanish since second fundamental form does not have corresponding component.

4. For CP_2 type extremals and cosmic strings Kähler gauge current vanishes. Self-duality of the induced Kähler form is one manner to satisfy field equations and in this case Kähler current vanishes. Vacuum extremals are an especially interesting type of extremals: also these could define algebraically analytic surfaces under some additional conditions.

4.4 Can the hyper-quaternionic solution ansatz be consistent with field equations associated with Kähler action?

The first thing to do is to check whether the properties of hyper-octonion analytic solution ansatz are consistent with $HO-H$ duality. The most obvious consistency requirements are satisfied. In H picture the field equations are formulated in terms of metric and Kähler form. The counterparts of metric and Kähler form should appear also in HO picture. At the level of HO the preferred hyper-octonion unit of HO would serve as the counterpart of the Kähler form. The decomposition of the tangent space of HO to hyper-quaternionic subspace and its complement requires the introduction of HO metric and the metric of X^4 induced from it. More precisely, the induction of $U(2)$ vector fields to HO and space-time surfaces requires index raising by HO metric and induced metric.

4.4.1 No hyper-quaternion analyticity at the level of $H = M^4 \times CP_2$

The basic question is whether hyper-octonion analyticity allows to express HO coordinates as hyper-quaternion analytic functions with real Laurent coefficients of a preferred hyper-quaternionic space-time coordinate h and whether the contraction of the energy momentum tensor and second fundamental form vanishes for each index pair separately in these coordinates.

The fact that the solution ansatz selection unique hyper-complex structure at each point of HO suggests that it is not possible to achieve hyper-quaternionic analyticity for $M^4 \times CP_2$. Also the fact that the Hyper Kähler structure of CP_2 fails to be covariantly constant supports the same conclusion.

This view is supported by the properties of rather general families of solutions of field equation constructed in [D1]. Hamilton-Jacobi coordinates are of form $(u, v, w, \overline{w})$ where u and v are light-like and $(w, \overline{w})$ complex coordinates for the decomposition $T(M^4) = M^2 \times E^2$ of the tangent space of M^4. CP_2 possesses also complex coordinates. At the level of HO this decomposition is replaced by the hyper-quaternionic coordinate $h = t + \sqrt{-1}(Iz + Jx + IJy)$. Since $\sqrt{-1}J$ corresponds to $\sqrt{-1}J_{xy}$, the Kähler form in the plane E^2, the relationship must be very intimate. known facts about the solutions of field equations.

4.4.2 HO and HQ analyticities as local symmetries?

HQ analyticity should be realized as a symmetry of the field equations associated with the Kähler action in the sense that HQ analytic deformations of the space-time surface define dynamical symmetries preserving the value of the action density. HQ analyticity is extremely strong constraint on dynamics, actually implying effective 2-dimensionality. HQ analyticity is in complete accordance with quantum criticality which is expected to give rise to a generalization of the 2-dimensional conformal invariance to a four-dimensional one. The conformal properties of the four-dimensional light cone boundary allow also a realization of complex super-conformal invariance: these two conformal symmetries are separate things.

HQ analytic transformations of the space-time coordinate give rise to mere coordinate transformations realizing conformal invariance as a gauge symmetry. The genuine deformations of the space-time surface represented by HQ analytic maps would in turn correspond to genuine dynamical conformal degrees of freedom. On the other hand, HO analytic maps of HO possibly generating new solutions to field equations would act also in the interior of the space-time surface non-trivially.

4.4.3 Hyper-octonionic analyticity and effective 2-dimensionality

The number of local integrability conditions is 6 corresponding to all index pairs for $U(2)$ algebra so that $8-6=2$ free functions should appear in the map g. The effective 2-dimensionality for the absolute minima is basic ideas of TGD and means that they are determined by the data at partonic 2-surfaces so that also this suggests algebraic two-dimensionality.

The effective 1-dimensionality due to the real analyticity of the hyper-octonionic map, would suggests that the ansatz is too limited. As a matter fact, g is expressible in terms of the conjugate of the hyper-octonionic map and its conjugate so that g depends on both h_T (that is $3\oplus\overline{3}$ of h) and its conjugate and in this the situation is algebraically 2-dimensional. That the longitudinal degrees of freedom corresponding to $(1, e_1)$ tangent plane do not appear in the expression for g, has physical interpretation in terms of the elimination of longitudinal polarizations. An analogous phenomenon occurs also for the known solutions and in Hamilton Jacobi coordinates $(u, v, w, \overline{w}$ for M^4 it corresponds to the plane spanned by the light-like coordinates u and v.

4.4.4 What is the dynamics of local HO automorphisms

The local G_2 element has interpretation as a local choice of the plane of un-physical polarizations or equivalently the plane of physical polarizations at space-time level. Also the interpretation as a local choice of quantization axis of angular momentum is possible.

The question is whether the local G_2 element f is completely free or whether the dynamics of Kähler action poses conditions on f.

1. If the choice is completely free, the action principle associated with HO dynamics would possess the dynamical analog of full local G_2 gauge invariance. This symmetry would become a dynamical symmetry in H picture, and could correspond to zero modes which do not reduce to gauge degrees of freedom although they do not quantum fluctuate (that is contribute to configuration space metric). A possible algebraic constraint is the commutativity $\hat{g}\circ\hat{f}=\hat{f}\circ\hat{g}$ implied by $f(h)=f(\overline{h}/|\overline{h}|)$. The freedom to perform (almost) completely free hyper-octonion automorphisms $f(h)$ would give hopes of understanding the vacuum degeneracy of the Kähler action.

2. There is a definite analogy with WZW model where a local Lie group element f defines the dynamical variable. In this case f is expressible in the form $f_L(z)f_R^{-1}(\overline{z})$ at 2-D boundary of 3-surfaces and continued to the interior in arbitrary manner. Conservation laws for Kähler action should however pose constraints on the G_2 local rotation of the imaginary part of h. This suggests that in case of non-vacuum extremals f obeys field equations of some kind in entire HO.

3. An attractive possibility is that ordinary analyticity of WZW case is replaced with hyper-octonion real-analyticity so that both g and f would be determined by the data at 2-D string like surface X_l^2 for which the tangent space at each point is spanned by real unit 1 and preferred imaginary unit e_1. This would mean effective 2-dimensionality. WZW action

with kinetic term restricted to this X^2_l and topological term restricted to a light-like causal determinant X^3_l suggests itself as the dynamics. Also hyper-octonionic Dirac action could be restricted to X^2_l. The outcome would be a reduction to an 8-D variant of string model.

4. The effective 2-dimensionality is a highly attractive notion but one must consider also other than hyper-octonion analytic continuations from X^2_l to HO, which might thus characterize only some particular solution types such as non-vacuum extremals or asymptotic solutions with vanishing Lorentz Kähler force and light-like Kähler current.

In fact, the existence of vacuum extremals requires more general local automorphisms than hyper-octonion analytic ones. If the map $f : HO \rightarrow S^6$ is such that the direction of the imaginary part depends on one parameter only, the image of HO in $SU(3)$ is 1-dimensional and CP_2 projection is also at most 1-dimensional. Vacuum extremals are obtained also when the $SU(3)$ image of HO is at least 4-dimensional and contains $U(2)$ subgroup: $X^4 = (g\circ f)^{-1}(U(2))$ would represent a vacuum extremal in this case. It is easy to imagine a situation in which f varies in such a manner that the foliation contains both vacuum extremals and non-vacuum extremals. An open question is whether the dimension of CP_2 image $g\circ f(X^3)$, $X^3 \subset X^4$ can vary from say 1 to 3 for a four-surface X^4 in a foliation defined by an arbitrary function f.

There is an interesting connection with the conjecture that S^6 does not allow complex structure although it allows almost complex structure which does not allow the representations of imaginary unit in local coordinates. Just before his death Chern published a proposal for a proof of this conjecture [37, 38]. Kähler structure is prerequisite for quantization so that the conjecture would be consistent with the idea that neither Kähler nor complex structure are possible in these degrees degrees of freedom so that local S^6 should indeed represent non-quantum fluctuating zero modes.

4.5 Spinors, calibrations, super-symmetries, and absolute minima of Kähler action

The proposed construction brings in mind the notion of calibration [20] and its connection with super-symmetries and minimal surfaces. There is a brief but very nice summary of this connection at the home page of Jose' Figueroa-O'Farrill [40]. Calibrations have been applied in a systematic study of branes [54, 42, 41].

4.5.1 Calibrations, minimal surfaces, spinors, and super-symmetries

The following arguments summarize the brief popular discussion, which can found at [40]. A more detailed source giving a lot of examples is the web article of Robert McLean [43] about deformations of calibration manifolds.

1. The notion of calibration

Let H be a manifold endowed with a Riemannian or pseudo Riemannian metric. A p-plane Π can be characterized by the p vectors e^a of H spanning Π as a vector space. Calibration in H can be defined as a closed p-form ω satisfying for each p-plane the condition

$$\omega(\Pi) \quad \leq \quad dvol(\Pi) \ . \tag{42}$$

Here the right hand side gives the volume of the p-plane in the metric of H, and is proportional to the square root of the determinant of the induced metric in p-plane. Π is said to be calibrated if the inequality is saturated:

$$\omega(\Pi) = \omega(e^1,...,e^p) \quad = \quad dvol(\Pi) \ . \tag{43}$$

A p-dimensional sub-manifold M is calibrated if the equality holds true for its tangent plane at each point. A fundamental result is that the manifolds satisfying this condition are minimal surfaces in a given homology class.

2. Examples

Since all known extremals of Kähler action with a non-vanishing action density are minimal surfaces calibrations are of obvious relevance also from TGD point of view. Some examples of calibrations are given in [43].

1. In 2m-dimensional Kähler manifold the exterior powers $J^p/p!$ of the Kähler form define 2p-dimensional calibrations and the minimal surfaces in question correspond to p-dimensional complex sub-manifolds. This example has direct relevance for TGD. String like objects correspond to surfaces $X^2 \times Y^2 \subset M^4 \times CP_2$ where X^2 and Y^2 are minimal surfaces. The possibility to identify Y^2 as a complex sub-manifold means that the locus of zeros for any polynomial $P(\xi_1, \xi_2)$ of CP_2 coordinates gives a solution of field equations. An essential point is that the Kähler current $\partial_\beta(J^{\alpha\beta}\sqrt{g})$ vanishes by the calibration property.

2. There are also calibrations which are possible only in special dimensions and of obvious interest in the recent case [43]. The associative calibration 3-form and co-associative calibration 4-form both defined in 7-dimensional manifold with holonomy group which is a subgroup of G_2. In recent case the decompositions $M^4 \times CP_2 = M^1 \times (E^3 \times CP_2)$ and $M^8 = M^1 \times E^7$ suggest that static minimal surfaces $X^4 = M^1 \times X^3$ could correspond to associative calibrations and space-like four-surfaces $X^4 \subset E^3 \times CP_2$ to co-associative calibrations. Cayley-calibration is 4-form define in 8-D manifold with holonomy contained in Spin(7) and also this calibration might be of relevance in TGD framework.

3. The connection with spinors

The connection with spinors comes as follows. The tensor products of a spinor field with its conjugate produce quite generally p-forms, which suggests that spinor fields could be used as building blocks of differential geometry. From covariantly constant spinors p-forms, which are also covariantly constant could be constructed. For instance, covariantly constant right handed neutrino in CP_2 gives rise to the CP_2 Kähler form. More generally, the solutions of Dirac equation produce closed p-forms.

Typically the representation of the calibration form in terms of spinor is obtained by considering covariantly constant spinor ϵ normalized to unity. The calibration form is of form

$$\omega = dx^{k_1} \wedge ....dx^{k_p}\bar{\epsilon}\gamma_{k_1}...\gamma_{k_p}\epsilon \qquad (44)$$

The obvious problem from TGD point of view is that for Minkowski signature of the metric of M^8 and chirality condition on ϵ implies identical vanishing of this kind of forms for $p = 4$. This problem is encountered also when octonionic 2-spinors with chiral condition are used. The way out of this problem is provided by introduction of preferred time direction defining gamma matrix added to the spinorial expectation value.

4. The connection with super-symmetry

The connection with the super-symmetry was discovered by Becker and Strominger [54]. In [44] examples about spinorial representations of calibrations in Minkowski space-time with one time like direction are discussed. The calibrations have dimension $p = 1, 2 \mod 4$ so that 4-dimensional calibrations of this kind are not possible. The calibration form is defined as

$$\omega = dx^{I_1} \wedge ... \wedge dx^{I_p} \epsilon^T \Gamma_{0I_1...I_p} \epsilon \ . \tag{45}$$

Here ϵ is covariantly constant real spinor normalized to unity ($\bar{\epsilon}\gamma^0\epsilon = \epsilon^T\epsilon = 1$) and $\Gamma_{0...I_p}$ is anti-symmetrized product of gamma matrices. Note that reality poses conditions on the dimension of Minkowski space.

For a given tangent plane Π the calibration is given by

$$\begin{aligned} \omega(\Pi) &= \sqrt{det(g)}\epsilon^T \Gamma(\Pi)\epsilon \ , \\ \Gamma(\Pi) &= \frac{1}{p!\sqrt{g}} \epsilon^{i_1...i_p} \Gamma_{0i_1...i_p} \ , \end{aligned} \tag{46}$$

where $\Gamma_{0i_1...i_p}$ is the projection of Γ to the plane Π in some coordinates labelled by $i_1, ..., i_p$. If the restriction on p is satisfied, the condition

$$\Gamma^2 = 1 \tag{47}$$

holds true and implies that ω defines a calibration. From this the alternative characterization for the saturation condition is as

$$\Gamma(\Pi)\epsilon = 1 \ . \tag{48}$$

The condition states that the minimal surfaces respecting the super-symmetry generated by covariantly constant spinor are minimal surfaces. What this means that the second order partial differential equations for minimal surfaces are replaced with eigenvalue condition for ϵ.

4.5.2 TGD based route to the connection between super-symmetry and minimal surface property

The connection between minimization of volume and super-symmetry emerges in TGD framework by different argument. The possibility to modify the Dirac action so that super-symmetry is not lost for Kähler suggests that also the notion of calibration might be generalized.

1. The Dirac equation for induced spinors in $X^4 \subset M^4 \times CP_2$ does not allow the covariantly constant right handed neutrino as a solution unless the trace of the second fundamental form vanishes which means that minimal surface equation is satisfied. Stated differently: the super-symmetry defined by the covariantly constant right handed neutrino remains super-symmetry at the space-time surface only if it is minimal surface.

2. The need to preserve the super-symmetry in the case of Kähler action led to the introduction of a modified Dirac action [B4] consistent also with the vacuum degeneracy of Kähler action. The modification works for any general coordinate invariant variational principle.

3. This observation raises the hope that the generalized calibration associated with an action defined by Kähler action density L_K could be defined as a closed 4-form ω_K constructed from the solution of the modified Dirac equation in H or of the hyper-quaternionic Dirac equation in HO.

The natural guess is that a 4-dimensional sub-manifold M is Kähler calibrated if the condition

$$\omega_K(\Pi) \quad = \quad L_K \times \omega(\Pi) = L_K \times dvol(\Pi) \tag{49}$$

holds true for its tangent planes. The generalized Kähler calibration form ω_K should be closed whereas now ω need not be closed anymore. Hence the definition is not equivalent with the standard calibration unless L_K reduces to constant different from zero. The generalization of the minimal surface condition would be that in a given homology class Kähler calibrated sub-manifolds are with a proper choice of the sign of action density minima of the action defined by L_K.

The above definition is of course only heuristic. The basic question is how the Kähler action associated with the space-time surface can appear in the calibration, which is defined in HO. The crucial idea is that the map $OH \rightarrow H$ assigns to each point of OH a hyper-quaternionic 4-plane allowing to assign the value of L_K with it.

The generalization of the connection with spinors would be of the same form and naturally based on the use of hyper-octonionic spinors which would also define the calibration. Note however that covariantly constant right handed neutrino spinor might make it possible to define calibration also in H.

4.5.3 Are asymptotic solutions of field equations with non-vanishing action density always minimal surfaces?

In the case of Kähler action the calibration condition would reduce to that for minimal surfaces when Kähler action density is non-vanishing and become trivial otherwise. This sounds rather counter-intuitive. However, all the known extremals with non-vanishing Kähler action are indeed minimal surfaces. The only known extremals which are not minimal surfaces are massless extremals and vacuum extremals for which Kähler action density indeed vanishes [D1].

These observations force to ask whether absolute minima/maxima would be common for a wide class of general coordinate invariant action principles. The conjecture is probably wrong. The reason is that for all known minimal surface solutions of field equations Kahler action density is constant. If Kähler calibration ω_K is expressible as a product $L_K\omega$, ω reduces to ordinary calibration when L_K is constant and minimal surfaces are obtained as solutions.

The field equations for Kähler action reduce to simple algebraic conditions for the known solutions. This of course could tell mostly only about my personal limitations. There could be however also deep physics involved. By quantum classical correspondence space-time surfaces should provide space-time correlates for asymptotic self-organization patterns, and are thus characterized by the vanishing of Lorentz Kähler 4-force. This requires that Kähler current vanishes or is light-like and the contraction of the energy momentum tensor with the second fundamental form vanishes: algebraically this condition is very similar to the minimal surface condition. This is achieved when energy momentum tensor and second fundamental form share no common index pairs. This is true also in the case of more general solutions such as massless extremals and vacuum extremals for which the condition fails for metric.

Asymptotic behavior might indeed correspond to the minimal surface equations or vanishing of Kähler action. That Kähler action would approach to constant is natural if one accepts the idea about space-time correlate for thermal equilibrium implying spatio-temporal homogeneity.

4.6 Number theoretic spontaneous compactification and calibrations

The question is whether number theoretic spontaneous compactification and the notion of Kähler calibration could be unified to a single description. The basic conceptual frameworks indeed share several common elements such as spinors and it turns out that the notion of Kähler calibration makes sense.

4.6.1 The notion of Kähler calibration

What is needed is a generalization of calibration so that minimal surfaces are replaced with the absolute minima of Kähler action. Hyper-quaternionicity condition is expected to be equivalent with the saturation condition for Kähler calibration.

The intuitive idea is that the non-closed 4-form ω, which is by definition saturated for the hyper-quaternionic planes, is not closed and does not therefore define minimal surfaces. ω becomes a closed form, when multiplied with Kähler action density L_K, and by its closedness defines absolute minima of Kähler action.

1. The map assigning a hyper-quaternionic plane to each point of M^8 defines a map $M^8 \rightarrow H = M^4 \times CP_2$, and this map can be used to induce the metric of H and Kähler form of CP_2 to M^8. The crucial point is that the image point in CP_2 allows to assign to the point of HO a hyper-quaternionic (or co-hyper-quaternionic) a 4-plane Π. The induced Kähler form J and its dual $\star J$ can be evaluated for Π, and one can assign Kähler action density $L_K = J \wedge \star J$ to each point of HO.

2. By index lowering operation using the induced metric in Π the contravariant 4-form $e^1 \wedge .. \wedge e^4$ defined by the unit tangent vectors of Π gives rise to a four-form ω in H. When evaluated for the hyper-quaternionic (or co-hyper-quaternionic) plane ω gives its volume. Saturation obviously occurs since the value of the form is in well defined sense the cosine for the angle between the plane Π and the hyper-quaternion plane.

3. The form ω is closed in general. The natural idea is that it becomes closed by multiplying it by a suitable integrating factor. The only natural candidate for the integrating factor is L_K. Thus Kähler calibration would correspond to a closed 4-form

$$\omega_K = L_K \omega \ , \tag{50}$$

which by construction is saturated by hyper-quaternionic (or co-hyper-quaternionic) 4-surfaces.

The hope is that the saturation is equivalent with the absolute minimization or at least extremization of Kähler action. Extremization is consistent with what is known about the extremals of Kähler action since L_K is constant for the known minimal surface extremals for which Kähler calibration reduces to ordinary calibration.

4. What characterizes the Kähler calibration is the map $HO \rightarrow M^4 \times CP_2$. Arbitrary maps are not allowed since $\omega_K = L_K \omega$ must be closed. Hyper-octonion real-analytic 2-component spinor fields Ψ of HO satisfying Weyl condition define in a natural manner maps $HO \rightarrow M^4 \times CP_2$. The conjecture is that these maps define Kähler calibrations which are by definition saturated by hyper-quaternionic or co-hyper-quaternionic manifolds.

4.6.2 Under what conditions extrema of Kähler action result?

Consider now the conditions under the construction gives extrema of Kähler action.

1. Suppose that Kähler action density has a positive value in a given region of a hyper-quaternionic 4-surface X^4. Closed-ness of ω_K implies that for local deformations restricted in this region the integral of the 4-form ω_K remains invariant. Since the value of ω is reduced from that for X^4, this means that $L_K \sqrt{g}$ increases in this region so that also Kähler action increases. On the other hand, if Kähler action density is negative, the value of Kähler action becomes even more negative in the deformation. Thus one might hope that absolute

minimum of Kähler action defined using $|L_K|$ as an action density has been found or that the variational principle using L_K as an action density minimizes the magnitudes of negative and positive contributions to the Kähler action separately.

2. There is however a hole in this argument. The point is that L_K is not varied at all at a given point of M^8 but fixed to its value for the selected hyper-quaternionic 4-plane. The manner to achieve a real minimum is to assume that the hyper-quaternionic plane at a given point provides a local minimum of the magnitude of L_K in the set of 4-planes at that point. The property of being an integrating factor should be consistent, and one might hope equivalent, with this property.

The basic question is why just the hyper-quaternionic 4-plane should minimize L_K, and why just L_K. The reader can decide whether or not to take with any seriousness the following linguistic musings. L_K is the only non-trivial action density defined by the map $M^8 \rightarrow M^4 \times CP_2$ ($J \wedge J$ would define a trivial action). The plane in question must be characterized by this map and CP_2 point indeed selects uniquely the hyper-quaternionic plane among other planes. The 4-form characterizing the plane and its spinorial counterpart should be expressible using octonionic algebra operations, and the possible non-uniqueness due to non-associativity must be avoided. Hyper-quaternionic plane and its dual are again the only natural candidates.

4.6.3 The number theoretic variational principle is not equivalent with the absolute minimization of Kähler action

There are two alternative identifications for the variational principle implied by the notion of Kähler calibration. Neither of them is consistent with absolute minimization of Kähler action.

1. The absolute minimization of the action defined by the absolute value $|L_K|$ of the Kähler action density is the first candidate for a variational principle. The problem is that the net action for the entire Universe is very probably infinite for this option, and the exponent of Kähler action infinite depending on the sign of Kähler function.

2. If L_K defines the action density, the number theoretic variational principle would minimize the magnitudes of positive and negative contributions to the action separately and be therefore very conservative. The action for the entire universe would tend to be small so that this option is strongly favored.

For both options Universe would do its best to save energy, being as near as possible to vacuum. Also vacuum extremals would become absolute minima: note that they would be only inertial vacua and carry non-vanishing density gravitational energy. The non-determinism of the vacuum extremals would have an interpretation in terms of the ability of Universe to engineer itself.

The 3-surfaces for which CP_2 projection is at least 2-dimensional and not a Lagrange manifold would correspond to non-vacua since conservation laws do not leave any other option. The variational principle would favor equally magnetic and electric configurations whereas absolute minimization of action based on L_K would favor electric configurations. The positive and negative contributions would be minimized for 4-surfaces in relative homology class since the boundary of X^4 defined by the intersections with 7-D light-like causal determinants would be fixed. Without this constraint only vacuum bubbles would result.

The attractiveness of the number theoretical variational principle from the point of calculability of TGD would be that the initial values for the time derivatives of the imbedding space coordinates at X^3 at light-like 7-D causal determinant could be computed by requiring that the energy of the solution is minimized. This could mean a computerizable solution to the construction of Kähler function.

4.6.4 Does a solution of hyper-octonionic Dirac equation define Kähler calibration?

Absolute minima of Kähler action should correspond in a dual manner to 4-surfaces in HO assigned to the hyper-octonion analytic solutions of the hyper-octonionic Dirac equation. The solution of the hyper-octonionic Dirac equation defines an element of the local $SU(3)$ Lie algebra and the exponentiation of this vector field defines map to $SU(3)$. The projection to CP_2 fixes the local hyper-quaternionic plane in HO.

The solutions of field equations defined by Kähler action should correspond to integrable distributions of these planes. This means that the tangent plane of the 4-surface coincides with the hyper-quaternionic plane at each point and the search for solutions would be very much like fine tuning (calibrating!) these planes. The guess is that the closedness of ω_K is enough to guarantee this and is implied by the hyper-octonion analyticity.

The induced metric and Kähler form appear naturally in the construction. The selection of the preferred complex plane (hyper-octonionic imaginary unit) naturally corresponds to the induction of CP_2 Kähler form to HO by the map $HO \rightarrow M^4 \times CP_2$ defined by the octonionic spinor field Ψ and by the canonical projection $M^8 \rightarrow M^4$. Also the metric of $M^4 \times CP_2$ can be induced to HO by this map. The consistency of the number theoretical metric signature with the signature of the induced metric favors hyper-octonions rather than octonions.

Since hyper-octonionic gamma matrices are used in the hyper-octonionic Dirac equation, induced metric does not appear at all in the Dirac equation involving only contravariant gamma matrices γ^k. Hence the solutions Ψ of the Dirac equation do not depend in any manner on the induced quantities. Induced metric appears only in the conditions defining the calibration and calibrated 4-surfaces.

If the tangent vectors e^i of the plane are regarded as hyper-octonionic vectors, and if the plane were not hyper-quaternionic, non-associativity could cause problems. For co-hyper-quaternionic planes the HO dual of ω is uniquely defined. Thus hyper-quaternionicity and its co-property seem to be forced by the internal consistency.

4.6.5 Co-hyper-quaternionicity and dual of Kähler calibration

One can argue that that the local minimization of L_K in the definition of Kähler calibration does not allow extremals with large Kähler action density, the most notable examples being CP_2 type extremals and cosmic strings. Therefore co-hyper-quaternionicity seems to be necessary.

One can imagine two options in co-hyper-quaternionic case.

1. The first option is that the local maximum (rather than minimum) of Kähler action density L_K defines the dual calibration. Dual Kähler calibration would maximize contrasts in the sense that the absolute value of the contribution to the action from a region with a fixed value of action density would be maximized. Stability considerations disfavor this option but quantum criticality characterized by large fluctuations favors it. Also the fact that Kähler action for CP_2 plays key role in TGD, favors this option. One might say that for this option universe and also configuration space of 3-surfaces would divide into disjoint regions corresponding to hawks and doves.

   Since there would be 3-surfaces which correspond to both Kähler function K and its dual K_D, uniqueness problems are encountered unless the Kähler functions are related by a transformation $K = K_D + \Phi(Z) + \overline{\Phi(Z)}$, where Φ is an analytic function of complex configuration space coordinates Z so that metric and Kähler form remain invariant.

2. An alternative option is the introduction of the dual of Kähler action defined by the normal projection of CP_2 Kähler form (this will be discussed later in more detail). One cannot exclude the equivalence of this option with a). This option would be conservative. Since

dual action vanishes for CP_2 for type extremals, this option would allow also them. The problem is that CP_2 action disappears from the theory. The same transformation as above must relate K and K_D.

4.7 Kähler calibration and spinor fields

Ordinary calibrations relate closely to spinors and one can wonder whether this is the case also now.

4.7.1 Does the spinorial equivalent for field equations exist?

The question is how to generalize the characterization of calibration in terms of covariantly constant spinor to the recent case. It is obvious that instead of Kähler calibration ω_K the non-closed form ω saturated for hyper-quaternionic 4-planes or their duals should be represented in terms of a spinor field normalized to unity as an "expectation value" of a suitable sigma matrix.

There are two options concerning the choice of the spinor.

Option 1: The covariantly constant right handed neutrino spinor of H, which is unique defined apart from a Lorentz rotation, is used.

Option 2: The octonionic spinor Ψ is used. In this case gamma matrices are 2×2 octonionic gamma matrices and non-associative. This would give strong motivation for why the only sensible calibrations are hyper-quaternionic or co-hyper quaternionic.

In both cases the problem is that in the 4-dimensional case the expectation value of an anti-symmetrized product of four gamma matrices for a spinor field of definite chirality vanishes identically. In order to obtain a non-vanishing result an additional gamma matrix must be introduced to the expectation value in a Lorentz invariant manner. The real octonion unit defines a preferred time like direction and the most natural identification of this time coordinate is as light cone proper time $a=\sqrt{m_{kl}m^km^l}$. The unit vector n_k defined by the gradient $\partial a/\partial m^k$ with respect to flat M^4 coordinates contracted with contra-variant gamma matrices would define gamma matrix $N\equiv\gamma^a=n_k\gamma^k$ satisfying $\gamma_a^2=g_{aa}=1$.

The calibration form ω is the wedge product of the 1-forms associated with the four unit vectors e^I defining the tangent vectors of the hyper-quaternionic plane:

$$\omega \quad = \quad e^1_{k_1}e^2_{k_2}e^3_{k_3}e^4_{k_4}dh^{k_1}\wedge ....\wedge dh^{k_4} \quad . \tag{51}$$

The anti-symmetrizations of the products $e^1_{k_1}e^2_{k_2}e^3_{k_3}e^4_{k_4}$ are expressible as expectation values of completely anti-symmetrized products of gamma matrices as

$$\begin{aligned}\epsilon^{k_1...k_4}e^1_{k_1}...e^4_{k_4} &= \overline{\psi}Nc^{l_1}_{k_1}....c^{l_4}_{k_4}\Gamma_{l_1...l_4}\psi \ , \\ N &= n_k\gamma^k \ .\end{aligned} \tag{52}$$

Here $\epsilon^{k_1k_2k_3k_4}$ is permutation symbol having values $\pm 1, 0$, c^l_k are some coefficients, and ψ is a spinor with unit norm: $\overline{\psi}N\psi=1$.

The value of ω for a general plane Π is

$$\begin{aligned}\omega(\Pi) &= \sqrt{g}\overline{\psi}N\Gamma(\Pi)\psi \ , \\ \Gamma(\Pi) &= \frac{1}{4!\sqrt{g}}\times\epsilon^{\alpha_1...\alpha_4}\times\frac{\partial(h^{k_1},...,h^{k_4})}{\partial(x^{\alpha_1},...,x^{\alpha_4})}\times c^{l_1}_{k_1}....c^{l_4}_{k_4}\times\Gamma_{l_1...l_4} \ .\end{aligned} \tag{53}$$

The condition $\Gamma^2(\Pi) = 1$ should hold true and would express the generalization of the identify $(n^k\gamma_k)^2 = 1$ satisfied by a unit vector generalized to the case of 4-plane. The expectation value has interpretation as cosine of the angle between Π and hyper-quaternionic plane. For the hyper-quaternionic planes the condition

$$\Gamma(\Pi)\psi = \psi \tag{54}$$

is satisfied by construction.

In the hyper-octonionic case gamma matrices are covariant hyper-octonionic 2×2-matrices. The lowering of the indices of contravariant gamma matrices is performed by using the metric induced from H to HO.

The unit spinor ψ should relate closely to the hyper-octonionic spinor field Ψ characterizing the distribution of hyper-quaternionic planes. A possible identification for the unit spinor ψ is as the imaginary part $Im(\Psi)$ of the octonion represented by Ψ normalized to unity. This identification is sensible everywhere except at the origin of HO. Also Ψ could be normalized to a unit octonion. The normalization is possible unless Ψ is light-like octonion. The association of singularities with light-like octonions would be very natural physically. The gamma matrix N would naturally correspond to the matrix γ_0.

The hyper-octonionic representation is definitely different from the ordinary Clifford algebra representation. The fact that calibrated planes are hyper-quaternionic means that non-associativity in the product of gamma matrices does not cause uniqueness problems in the definition of ω using Ψ if the associativity of the tangent vectors is equivalent with the associativity of the corresponding 1-forms. Hence it seems that purely number theoretic constraints give for hyper-quaternionic and co-hyper-quaternionic calibrations a unique status.

4.7.2 Could the solutions of hyper-octonionic Dirac equation define a foliation of solutions of the modified Dirac equation?

Super-symmetry inspires the question whether the modified Dirac equation associated with Kähler action could be satisfied by Ψ restricted to X^4. This conjecture would mean that the theory could be solved both in fermionic and bosonic sector by the same general solution ansatz. This correspondence makes only sense if it is possible to map the induced hyper-octonionic gamma matrices to the gamma matrices induced from H. Hyper-quaternionicity implies associativity and raises the hope that the induced hyper-octonionic gamma matrices could be replaced with the gamma matrices of the imbedding space.

Note however that this ansatz would give only single solution of the modified Dirac equation in the interior of space-time surface unless there is large number of hyper-octonionic spinor fields containing the same space-time surface in the foliations that they define. A possible physical interpretation would be that the spinor field in the interior couples to the surface dynamics and is determined by it.

5 How $HO - H$ duality could be realized at quantum level of quantum TGD?

The classical $HO - H$ duality inspires several questions.

1. Does HO picture generalize to quantum level? Should one quantize hyper-octonionic spinors in some manner consistent with the identification of the space-time surface as a hyper-quaternionic 4-surface and the requirement that representations of quantum states result in this manner?

2. Are HO and H pictures dual and equivalent representations of the dynamics or are they pieces of a bigger structure needed to understand quantum dynamics.

   i) Strict duality brings in mind the somewhat fuzzy notion of 7-3 duality introduced earlier [A2, C1]. Strict duality would mean the dynamics of appropriately quantized hyper-octonionic spinor fields at space-time interior would be a dual counterpart of the dynamics of second quantized free spinor fields of H at 3-D light-like CDs.

   ii) One can also defend the idea that both H and HO dynamics are needed to obtain a complete description of dynamics. Since HO dynamics dictates the dynamics of 4-surfaces, only the spinorial dynamics associated with 3-D light-like CDs would represent H dynamics not reducible to HO dynamics in the proposed form.

5.1 Only quantized octonionic spinors fields could be consistent with $HO-H$ duality

A simple argument demonstrates that $H-HO$ duality is possible at quantum level only if octonionic spinor fields are quantized. The requirement that the quantization is consistent with the identification of space-time surfaces as hyper-quaternionic 4-surfaces fixes the quantization and provides a connection with quantum measurement theory.

5.1.1 Classical hyper-octonionic spinor fields cannot give rise to $HO-H$ duality at quantum level

It is easy to get convinced that classical hyper-octonionic spinor fields cannot give rise to $HO-H$ duality at quantum level. The hyper-octonion analytic functions involve local G_2 rotation of the imaginary part of hyper-octonion. The directions of the imaginary part of $O_h(h)$ correspond to points in S^6. $\delta M^8_\pm$ and, more naturally, $\delta M^4_\pm \times E^4$, represent the candidates for the counterpart of 7-D light-like causal determinant (CD) in HO. The restriction to $\delta M^8_\pm$ ($\delta M^4_\pm \times E^4$) means metrically a restriction to S^6 ($S^2 \times E^4$) so that a map from S^6 ($S^2 \times E^4$) to S^6 parameterizes the local selection of the plane of non-physical polarizations. These local degrees of freedom correspond to zero modes rather than quantum fluctuating degrees of freedom.

The metric 6-dimensionality of $\delta M^4_+ \times CP_2$ conforms with the view that the selection of the plane of non-physical polarizations in HO leaves 6 local polarization directions. Super-Kac-Moody and super-canonical degrees of freedom of metrically 6-D $\delta M^4_+ \times CP_2$ correspond to these degrees of freedom naturally.

If hyper-octonionic spinor fields are classical fields, the restriction or possibly residue of the hyper-analytic spinor field at 7-D light-like CD $X^7 \subset HO$ selects only the local plane of non-physical polarizations at X^7, and hyper-octonion analytic degrees of freedom at X^7 cannot correspond to super Kac-Moody and super-canonical degrees of freedom.

5.1.2 Real-analytic HO spinor fields as zero modes of HO spinor fields

Classical hyper-octonionic spinor fields cannot code for quantum states nor can they define gamma matrices of the configuration space CHO of 3-surfaces $X^3 \subset HO$. One can imagine two solutions of the problem.

H spinor fields induced to $X^4 \subset H$ and satisfying the modified Dirac equation are interpreted as zero modes representing super-symmetries and are not second quantized. This suggests that classical hyper-octonionic spinor fields represent zero modes of HO spinor fields just as the solutions of the modified Dirac equation represent zero modes of induced spinor fields in H. The second quantized part of induced HO spinor fields at $X^4 \subset HO$ would define the gamma matrices of configuration space CHO acting as super generators. This is certainly what must happen if $HO - H$ duality is realized at the level of configuration space.

Also the second quantized fields would have $1 + 1 + 3 + \overline{3}$ decomposition having interpretation in terms of leptons and quarks. The fermionic oscillator operator valued Laurent coefficients of the quantized hyper-octonionic spinor field would commute with the hyper-octonionic units and have interpretation in terms of leptonic and quark like creation operators and anti-leptonic anti-quark like annihilation operators: this allows to achieve the conservation of lepton and quark numbers.

Strict $HO - H$ duality requires however more than this. Full $HO - H$ duality would suggest that

i) second quantization occurs for induced spinor fields at $X^4 \subset HO$ ($X^4 \subset H$) but it is an open question whether and in what sense it occurs for spinor fields in HO (H),

ii) also the solutions of massless Dirac equation in H play a role analogous to that played by real-analytic spinor fields in HO and define the same foliation of HO as real-analytic HO spinor fields.

It would be awkward to introduce X^4 and H (HO) spinor fields as independent dynamical degrees of freedom so that there should be a relation between them. The simplest relation is that the induced spinor fields satisfying the modified Dirac equation at the 4-surfaces of the foliation can be regarded as restrictions of H (HO) spinor fields satisfying the massless Dirac equation in H (HO). This could hold true not only for the zero mode part but also for the second quantized part of these spinor fields. This point will be discussed in last section of the chapter.

5.1.3 Do HO spinor fields provide a representation for observables characterizing quantum state as a space-time surface?

In some sense HO spinor fields or at least the induced HO spinor fields must be quantized. The original idea was that hyper-octonionic spinor field is quantized so that its Laurent coefficients become mutually commuting Hermitian operators acting in the representation space of super-Kac-Moody and super-canonical algebras. Although this quantization is not necessary and not enough to construct configuration space spinor structure in CHO, it could quite neatly realize quantum classical correspondence. Of course, one must be cautious in introducing any kind of extra structure but the idea is so beautiful that it deserves a discussion.

Also the real matrix elements of the matrix representing HO-local G_2 rotation would be expressible in terms of commuting Hermitian operators. The Hermitian matrix elements associated with different points of HO need not commute. Physical intuition however suggests that they commute for points in a maximal deterministic region of a given space-time sheet since maximal deterministic regions represent final states of quantum jumps classically.

The commutativity at different points of a maximal deterministic region of space-time sheet could be interpreted as existence of a wave function, and would be therefore a number theoretic counterpart for quantum coherence. Thus hyper-octonion spinor field would be more like Schrödinger amplitude coding for quantum states via its Laurent series than quantum field. There is a definite analogy with a foliation of $2n$-dimensional phase space by Lagrangian manifolds by n-dimensional surfaces at which wave functions are defined in geometric quantization.

A one-one correspondence with quantum states is achieved if the Hermitian operators in question represent a complete set of observables allowing to fix uniquely the physical state. In this context the hierarchy of infinite primes represented as hyper-octonion analytic polynomials [E3] would correspond to a hierarchy in which the number of mutually commuting observables would increase. The requirement that the eigenvalues of operator coefficients are consistent with the interpretation in terms of infinite primes gives constraints on the choice of observables. A representational triad consisting of quantum states, space-time surfaces, and infinite primes emerges.

This picture is also consistent with the general ideas about state function reduction as a localization in the zero modes interpreted as non-quantum fluctuating parameters labelling 3-surfaces.

The 4-D parameter space P labelling the (hyper-quaternionic) maximal deterministic regions in the foliation corresponds to zero modes. Quantum superpositions of these regions correspond to wave functions in P. The Hermitian operators representing G_2 matrix elements at maximal deterministic regions corresponding to different points of P do not commute in general. Hence state function reduction forces a complete localization in P, and the final state is completely classical, single space-time surface. The matrix element of the inner product between a P-non-local state resulting in "U-process" and a P-localized, completely classical state resulting in the state function reduction is well-defined and finite.

The quantization of the notion of space-time as a hyper-quaternionic surface adds a further candidate to the long list of non-commutative/quantum geometries. Space-time could be well defined classically only in eigen-states of the mutually commuting Hermitian coefficients of the hyper-octonionic power series, and this would force the map of quantum states to space-time surfaces.

This interpretation would also provide a further perspective to the notion of quantum classical correspondence by assigning a foliation of space-time surfaces to given eigenvalues of observables and representing a local selection of non-physical polarization directions representing the selection of quantization axis of spin at space-time level. Space-time or rather foliation of HO by space-time surfaces would be an "emergent" phenomenon providing a representation for quantum state.

A word of criticism is of course in order. The problem of how the observables are represented by the Laurent coefficients of HO spinor fields have not been discussed at all. If there exists no simple answer to this problem, the proposed quantization remains a nice looking but un-necessary branch in the tree of TGD.

5.2 Universal expressions for vertices using $HO-H$ duality?

The dynamics of generalized Feynman diagrams involves propagation and vertices at which space-time sheets are glued together along their ends. The challenge is to code this dynamics into the dynamics of hyper-octonionic spinor fields. Sometimes it is a good idea to be really brave. So, let us ask how simple the HO description of generalized Feynman diagrams could be at HO level. The answer is amazing in its generality: octonionic inner product and product, or equivalently duality and triality, provide universal expressions for 2-vertices and n-vertices in HO picture!

This prescription was originally proposed for real HO spinor fields with Hermitian Laurent coefficients and the following consideration are restricted to this case.

5.2.1 Propagation and hyper-octonionic inner product

The octonionic part of the inner product for hyper-octonionic spinors provides a good candidate for S-matrix elements describing internal transitions of particle understood in an extremely general sense. A Feynman diagrammatic interpretation of the single particle S-matrix elements is in terms of 2-vertices appearing naturally in the approach in which S-matrix results as a generalization of the unitary S-matrix associated with braidings [E9, C5].

1. Quantum classical correspondence implies that maximal deterministic regions correspond to space-time correlates of the final states of quantum jumps. In H picture 3-D light-like causal determinants at which the non-determinism of quantum jumps is located code for the dynamics. This could be true also in HO picture although one cannot exclude 3-D space-like causal determinants from the consideration.

2. Suppose that $HO-H$ duality means that interior dynamics of HO hyper-octonionic spinors codes for the quantum evolution at space-time level. This means that all details about partonic description are coded implicitly to the mutually commuting Hermitian Laurent coefficients of G_2 matrix elements and of HO analytic function.

3. Assume that the convergence regions for the power series defining G_2 matrix elements and hyper-octonionic Taylor series correspond to maximal deterministic regions. Assume the commutativity of Hermitian operators representing G_2 rotations inside these regions. At the boundaries of these regions G_2 element is discontinuous. What these assumptions state is essentially quantum coherence in generalized sense.

4. p-Adic topology allows description of non-determinism in terms of p-adic pseudo-constants. For some value of p p-adic topology is an excellent candidate for the effective topology in which these discontinuities are not discontinuities anymore. p-Adic fractality would characterize the breaking of quantum coherence.

5. Hyper-octonionic spinor field allows a natural inner product defined as an integral of the overlap of the spinor field at the two sides of discontinuity characterized by 3-dimensional CD. These integrals might actually reduce to integrals over 1-D string like surfaces. 2-component spinors with Weyl condition certainly suggest this. Also hyper-octonion analyticity, if it generalizes to the case of local G_2 rotation, means that the information characterizing hyper-octonionic spinor field is coded by 1-D curves.

6. If hyper-octonionic spinor fields code for quantum states, it is natural to assume that this inner product codes for the S-matrix element for the transition between initial and final internal states determined by the discontinuities of G_2 matrix element and hyper-octonion analytic power series. The value of this inner product, call it S_{mn}, is in general a complexified octonion. The octonionic part of S_{mn} can be interpreted as a complex number with the direction of the imaginary part determining the identification of the imaginary unit. S_{mn} would have interpretation as the S-matrix element between the initial and final internal states of the particle.

5.2.2 Interaction vertices and generalized Feynman diagrams as computations

The generalized Feynman diagrams differ from braid diagrams by the presence of $n > 2$ vertices. In the generalized vertices $n > 2$ space-time sheets are glued together along their ends just like the ends of lines in the vertex of an ordinary Feynman diagram [C5]. The challenge is to identify the HO counterpart of the vertex. According to the considerations of [C5] generalized Feynman diagrams can be regarded as being analogous to computations so that the equivalence of diagrams with loops to tree diagrams means that these diagrams represent equivalent computations. The requirement that generalized diagrams are equivalent with tree diagrams implies also that the lines of tree diagram can be contracted to point so that single n+m vertex remains.

Taking the computational argument very seriously, one can argue that the vertices must be constructible using hyper-octonionic product and inner product. The octonionic 2-spinors satisfying Weyl condition are in 1-1 correspondence with octonions so that product and inner product for hyper-octonionic spinors reduce to those for hyper-octonions. Apart from the constraints due to statistics there are no explicit constraints related to conservation laws since octonions are not regarded as representations of Spin(8). Of course, the map of quantum states to hyper-octonionic spinors must take into account these constraints.

The natural guess is that the vertex describing a reaction with n incoming states and n outgoing states reduces to the point-wise inner product of the point-wise product of m incoming octonion spinor fields and of n outgoing octonionic spinor fields appropriately symmetrized or antisymmetrized in order to take the statistics into account. This point-wise inner product, when integrated over the 3-surface defining the vertex using the volume element defined by the induced metric, would give the S-matrix element. The complications due to the non-commutativity and non-associativity should disappear by the symmetrization/anti-symmetrization required by statistics. The inner product is crossing symmetric.

The observables representing the quantum numbers of incoming and outgoing state and appearing as Laurent coefficients need not commute. If it is the eigenvalues of these observables, which occur in the inner product, vertex is not changed in the unitary transformations of a complete set of observables of a given incoming line induced by, say, rotations. Thus it seems that one must express all incoming and outgoing states using the same eigen basis so that the matrices of the unitary transformations appear in the vertices. Situation changes if the observables associated with different maximal deterministic regions commute. That the selection of observables would be same for the entire space-time surface rather than inside single maximal deterministic region or space-time sheet, seems an unrealistic assumption.

Later it will be found that the local G_2 element need not be completely free but could be subject to dynamics and expressible as product $g = g_L(h)g_R^{-1}(\overline{h})$ just like the solutions of field equations in WZW conformal field theory. Since the information about hyper-octonionic spinor field would be restricted on "string orbits" located most naturally on light-like CDs X_l^3, this would mean that string orbits in HO would code for the whole 4-D theory apart from corrections from the configuration space integration requiring the knowledge of Kähler function (fixed also by 2-D dynamics)! In particular, the calculation of S-matrix elements would reduce to the calculation of inner products of products of hyper-octonionic inner products at 2-dimensional string orbits.

5.2.3 Trialities and TOEs

Three is the sacred number of mystics and religions and it seems that this number is fundamental also from the point of physics. Indeed, the notion of triality [29] is fundamental for the mathematics of both string models and as it seems, also of TGD. Triality is a non-degenerate trilinear map $t : V_1 \times V_2 \times V_3 \rightarrow R$ of three vector spaces isomorphic to each other. A normed triality satisfies the bound $t(a, b, c) \leq 1$ and for given a and b inequality is always saturated by some choice of c. Trialities are very rare and it can be shown that they correspond to the classical division algebras. Triality can be expressed in terms of the operations of the division algebra as $t(a, b, c) = Re(\bar{c}ab)$, and is symmetric with respect to the cyclic permutations of its arguments.

Trialities relate closely to spinors and Clifford algebras [29]. The existence of triality corresponds to the fact that Clifford algebras in dimensions $n = 3$ and $n = 7$ decompose to a sum of two reducible real representations of $Spin(n + 1)$ having dimension $n + 1$ corresponding to two chiralities of Majorana spinors. Triality can be identified as the map $V_n \times S_n^+ \times S_n^- \rightarrow R$, where V_n corresponds to $n+1$-dimensional vector representation of $Spin(n+1)$ acting on $n+1$- component Majorana-Weyl spinors of positive and negative chirality as gamma matrices (note the restriction $n = 3, 7$). The automorphism group of the octonionic triality is $Spin(8)$, the Lie group with the most symmetric Dynkin diagram, for which the three outer nodes of diagram correspond to the 8-dimensional vector representation and left and right handed spinor representations of Spin(8).

Triality is an excellent candidate for defining fundamental coupling structures. The basic problem is that the octonionic norm is Euclidian. In string models this forces to increase of space-time dimension to 10 by replacing E^8 with $M(1, 9)$. The crucial observation is that $SL(2, O)$, which is isomorphic to $SO(1, 9)$ acts in a natural manner in O^2 just like $SL(2, C)$ acts in the space of 2-component complex spinors.

In TGD framework the introduction of hyper-octonions provides the means for overcoming the difficulties caused by the Minkowskian signature. By linearity the octonionic triality $t(h_1, h_2, h_3)$ generalizes without difficulties to the hyper-octonionic case. The replacement of real spinors with hyper-octonionic 2-spinors is of course also essential. A further element is the new interpretation inspired by $HO - H$ duality.

The appropriately symmetrized triality of three hyper-octonionic spinor fields integrated over the 3-surface defining the 3-vertex suggest a universal number theoretic coding of the generalized 3-vertex. Higher vertices are obtained by forming composite functions in which the map $(h_1, h_2) \rightarrow \overline{h}_3 = h_1 h_2$ defined by triality with the help of duality defined by the inner product is used. This boils down to the straightforward formation of octonion products for incoming and outgoing states. Thus duality would define propagation and triality would determine interaction vertices and thus also coupling constants at extremely general level. Hence the idea about correspondence between algebra operations and particle interactions developed in [C5] to an axiomatic form using the notion of bi-algebras would be realized very concretely in HO picture.

5.2.4 Number theoretic construction of vertices fails for second quantized parts of HO spinor fields

The previous arguments were based on the classical part of HO spinor field and on strong quantum-classical correspondence allowing to express S-matrix elements in terms of overlaps of the classical quantities.

The prescription does not have a natural generalization to the case of second quantized HO spinor fields having $1+1+3+\overline{3}$ decomposition in terms of hyper-octonionic imaginary units and fermionic oscillator operator valued coefficients commuting with them. The obvious reason is that fermion number conservation allows only vertices with an even number of lines and forces a restriction to the components of second quantized HO spinor field with a well defined fermion number.

If number theoretic prescription works, it allows a dual coding of S-matrix in terms of classical data alone. Classical real-analytic HO spinor field would map to a given 3-surface a unique quantum state whereas configuration space spin degrees of freedom assign an infinite number of different states to a given 3-surface. The paradox might be however avoided. According to TGD inspired quantum measurement theory, the values of configuration space zero modes and spinorial zero modes represent classical outcomes of quantum measurements in 1-1 correlation with quantum states in a given basis. The variation of an element of quantum state basis associated with a given 3-surface would affect the zero modes but leave configuration space metric invariant apart from possible conformal scaling depending on zero modes.

5.3 Does HO picture reduce to 8-D WZW string model?

The idea that 8-D WZW string model could determine the dynamics of non-vacuum extremals at both classical and quantum level in HO picture is so far-reaching in its simplicity that it deserves a serious discussion.

5.3.1 Could WZW action and hyper-octonionic Dirac action reduce the dynamics of the hyper-octonionic spinor fields to 8-D string model?

The local G_2 rotation of the imaginary part of HO coordinate need not be completely free, and the most obvious dynamics is the generalization of the dynamics defined by Wess-Zumino-Witten action to the hyper-octonionic context.

1. The analog of WZW action associated with G_2 valued chiral field $g(h)$ in HO would define in a very natural manner a TQFT as a conformal field theory. G_2 valued chiral field would be quantized in the sense that the Laurent coefficients of the matrix elements of g are Hermitian operators representing observables coding for the physical states.

2. The generalization of the complex analyticity to hyper-octonion analyticity is highly suggestive. A good guess is that the preferred 3-surfaces correspond to the time-like 3-surfaces X^3 at which space-time surfaces are branched. Preferred 2-surfaces would in turn correspond to the 2-dimensional hyper-complex time-like sub-manifolds X^2 of X^3 for which the tangent space is spanned by 1 and the preferred hyper-octonionic unit e_1. These surfaces would be dual to the partonic 2-surfaces (note that corresponding space-time surface correspond

hyper-quaternionic and co-hyper-quaternionic 4-surfaces intersecting generically in point-wise manner). This would give a deeper meaning for the choice of the preferred imaginary unit. The TQFT would be associated with the light-like 3-surfaces X_l^3 defining generalized Feynman diagrams in accordance with the TGD inspired view about topological quantum computation [E9].

3. The counterpart of the kinetic term of WZW action would be assigned with string orbit X^2 and topological term with X^3. Also the Dirac action for hyper-octonionic spinor field could be restricted to X^2 since all relevant information about the field is coded by Laurent expansion at this surface. Hence the local G_2 element would be obtained by continuing hyper-complex analytic solutions from X^2 to HO and would be expressible as $g(h) = g_L(h)g_R^{-1}(\overline{h})$ in chiral decomposition.

4. To gain some understanding about what might be involved it is good to summarize the H picture, which is based on partons rather than string orbits. H picture makes sense for co-hyper-quaternionic 4-surfaces, which cannot contain stringy 2-surfaces and the local choice of the preferred hyper-octonionic imaginary unit is not integrable so that string orbit does not exist. In H picture the construction would reduce by the effective 2-dimensionality to the construction of correlation functions at space-like partonic 2-surfaces interpreted as space-like surfaces $X^2 = X_l^3 \cap X^3$, where X^3 is the space-like 3-surface at which the ends of space-time sheets meet like pages of book and X_l^3 is light-like CD. In this case $G_2/SU(3)$ WZW model with octonionic spinors does not make sense and is replaced by the modified Dirac action for second quantized free induced H spinor fields (whether $SU(3)/U(2)$ WZW action is needed is not quite clear). This is in accordance with the fact that in H picture electro-weak and ordinary spin are spin like quantum numbers instead of color.

5. In HO picture space-like partonic 2-surfaces are replaced by time-like string orbits and string model in HO would become a part of TGD in the sense that hyper-octonionic spinor field would be obtained by an analytic continuation from this surface, and in turn define the space-time surface. Even the construction of S-matrix at space-time level using HO picture should reduce to a some kind of string model consistent with duality. Duality allows to translate H picture to HO picture.

Suppose that 4-surfaces can meet (branch) also along time-like 3-surfaces X^3. These 4-surfaces can contain 3-D causal light-like determinants X_l^3 (CDs). String orbits would correspond naturally to the 2-dimensional intersections $X^3 \cap X_l^3$. Hyper-octonion analyticity would allow to code generalized Feynman diagrams to generalized string diagrams and a rather close relation with string model amplitudes is expected. In particular, the poorly defined path integral over string orbits would be replaced by a well-defined functional integral defined by the dual of the Kähler function. The physical interpretation would of course differ dramatically from that in string models since string orbits would appear as intersections of higher-dimensional objects and would correspond to physical configurations rather paths connecting them. Therefore functional integral defining loop summation would not appear in the formalism.

Together with the identification of vertices in terms of local hyper-octonionic products this would mean an enormous simplification. Only the functional integral over the quantum fluctuations at configuration space level would require information about space-time surfaces, in particular about the value of the dual of the Kähler function and its second variation.

5.3.2 Does WZW action define the topological field theory associated with TGD?

The attempt to understand the dynamics of string models has led to the notion of topological string models [45, 46]. One can ask whether also "topological TGD" could exist. The TGD inspired vision about topological quantum computation [E9] indeed leads to the idea that a topological quantum field theory counterpart of TGD should exist. The fact that the generalized Feynman diagrams are obtained from braid diagrams by allowing branchings suggests that the TQFT in question could define besides knot-, link-, and 3-manifold invariants also more general invariants assignable with the generalized Feynman diagrams. WZW action appearing naturally in the construction of HO variant of TGD provides indeed a natural candidate for TQFT associated with TGD.

1. Chern-Simons action and topological QFTs

In the seminal work of Witten [49] the functional integral defined by the exponent of Chern-Simons action for gauge fields in group G defines knot-, link-, and 3-manifold invariants. The action density is given by

$$L_{CS} = \frac{k}{4\pi} Tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \ ,$$

where k is integer. The invariants are defined formally as vacuum expectations of products of Wilson loops defined as traces of non-integrable phase factors for the loops for the components of the link. The components of the link can corresponds to different irreducible representations of G.

The perturbative calculation of the invariants is based on saddle point approximation around pure gauge configurations and gives an infinite series of perturbative link invariants. Witten derives information about the values of the invariants non-perturbatively using ingenious arguments, and in some cases the link invariants can be calculated recursively using the so called skein relations.

2. WZW action and topological QFT

G_2 connections are pure gauge whereas for C-S model only classical solutions are pure gauge. The following argument strengthens the belief that the difficulty is only apparent.

1. Classical field equations $dA + [A, A] = 0$ state that A are pure gauge. This condition holds true in the functional integral approach in the sense that the vacuum expectation value for the first variation of action vanishes. In the Hamiltonian quantization the pure gauge property should hold true in a quantal sense.

2. The moduli space for the pure gauge connections is parameterized by the holonomies over the homotopically non-trivial curves of X^3. This space corresponds to the space of highest weight states for the Wess-Zumino-Witten model, which is conformal field theory at the boundaries of X^3. The dynamical variable is G-valued conformal field $g(x)$, and WZW action consists of kinetic terms at the boundaries of 3-manifold and topological interior term identifiable as a Chern-Simons term for a pure gauge field defined by $g(x)$.

   The functional averages of Wilson loops can be expressed in terms of correlation functions of a conformal field theory based on Wess-Zumino-Witten action having G as a target space. The flatness of quantized connections is indeed consistent with the fact that quantum equations of motion state the flatness of the connection in case of Chern-Simons action. Physical intuition suggests that vacuum expectation values for Wilson loops correspond to "thermal" expectation values for the WZW model defined by WZW-action in the sub-space of the state space defined by highest weight vectors of the unitary representations of corresponding Kac-Moody group and identifiable as the moduli space of flat G connections. Note that effective 2-dimensionality is realized now in topological sense.

3. Only pure gauge connections are needed WZW model. The field equations associated with WZW action imply that g at 2-D boundaries is of form $g_l(z)g_r^{-1}(\overline{z})$ in complex coordinates.

3. Could topological quantum field theory give information about physics?

Quantum-classical correspondence suggest also a more real-physics related approach to topological quantum field theory.

1. By previous arguments one expects that the functional integral average for the product of Wilson loops using Chern-Simons action should be replaced with a functional average using WZW action with quantized g and that links are replaced with generalized braids with the ends of the braids at incoming and outgoing partonic boundary components or more general branched braids associated with Feynman diagrams. The lines of braid are replaced with strings/ribbons as is done also in TQFT:s in order to describe rigorously self linkage.

2. The lack of a rigorous definition of the functional integral is a problematic aspect of topological quantum field theories. Since physical states correspond to G_2 valued chiral fields, one might hope that the functional expectation value using WZW action could be translated to average over physical states. The degeneracy factor multiplying the product of Wilson loops for given values of moduli would be the number of physical states consistent with the 3-topology and the values of moduli. The outcome would be automatically a topological and conformal invariant.

3. If these topological invariants are same as deduced by indirect arguments for G_2 Chern-Simons action, they would give a valuable information about the degeneracies of physical states with given values of G_2 moduli and fixed 3-topology. The correlation functions of G_2 WZW model might also define a natural starting point for a perturbative approach to the dynamics in non-topological degrees of freedom.

4. $HO - H$ duality forces to ask whether the ground states of G_2 Kac-Moody representations could be in 1-1 correspondence with the ground states of the representations of super-canonical and super Kac-Moody algebra. States in G_2 representations are labelled by color quantum numbers so that the correspondence is not excluded. If G_2 degrees of freedom correspond to zero modes this duality could be interpreted in terms of duality between quantum fluctuating degrees of freedom and zero modes representing the corresponding classical variables.

5.3.3 $G_2/SU(3)$ coset theory and QCD

Besides WZW models so called minimal models are of special practical significance since they are unitary, the number of the primary fields is finite for them, and the correlation functions satisfy an infinite number of differential equations and are calculable. WZW models define minimal models via coset construction [56]. $G_2/SU(3)$ coset model is for obvious physical reasons of a special interest now.

The G/H coset construction for a WZW model based on group G means averaging over H degrees of freedom by introducing an action term associated with H gauge field and functionally integrating over these degrees of freedom [56]. The correlation functions can be constructed exactly. At the level of Virasoro algebra it means the construction of Virasoro algebra generators as differences $L_n(G) - L_n(H)$. In the special case $G = H$ a topological field theory results.

At the level of highest level representations the coset construction means following.

1. The highest weight representations $\mathcal{H}^g_\lambda$ for G can be decomposed to the HW representations $\mathcal{H}^h_\lambda$ of H as

$$\mathcal{H}^g_\lambda = \mathcal{H}^\lambda_{\lambda'} \otimes \mathcal{H}^h_{\lambda'} \ . \tag{55}$$

In this decomposition $\mathcal{H}_{\lambda'}^{\lambda}$ is so called branching space, whose dimension tells how many times the HW representation λ' of H occurs in the HW representation λ of G the dimension. One can wonder whether the branching spaces for $G_2/SU(3)$ coset model could allow an identification in terms to electro-weak degrees of freedom. This is not the case since the branching spaces turn out to be one-dimensional.

2. The physical states of the coset theory are annihilated by the Kac-Moody generators $J_n^{H,a}$, $n>0$, of H

$$J_n^{H,a}|v\rangle = 0 \ , \quad n>0 \ . \tag{56}$$

This means that only the finite-dimensional representations of H defined by the highest weighs of H appearing in the decomposition remain in the spectrum. These conditions are equivalent with the conditions stating local $SU(3)$ gauge invariance so that the natural question is whether $G_2/SU(3)$ coset theory is only a topological field theory in $SU(3)$ degrees of freedom or whether the coset model is forced physically instead of the full WZW theory.

For the latter option the number theoretical interpretation for QCD would be that $SU(3)$ leaving invariant the choice of the preferred octonionic imaginary unit e_1 acts as a gauge group. Furthermore, it is $S^6 = G_2/SU(3)$ which codes for the choice of the preferred imaginary octonionic unit e_1 so that $G_2/SU(3)$ coset model is indeed natural. This also means that $SU(3)$ WZW model for which primary fields correspond to singlet, triplet and anti-triplet is not acceptable.

The condition $c<1$ fixes the value of $k(SU(3))$ to $k(SU(3)) = 1$ and the minimal model in question would have

$$c = c(G_2, k=1) - c(SU(3), k=1) = \frac{4}{5} \ . \tag{57}$$

$c = 4/5$ corresponds to a rational conformal field theory with 20 primary fields on basis of Kac formula. The number of primary fields reduces to 12 for three-state Potts model [57], which has been proposed as a model for 2+1-dimensional QCD [58]. For $c = 4/5$ the Virasoro algebra represented by quadratic color Casimir extends to W_3 algebra containing third order color Casimir as a primary field of conformal weight $\Delta = 3$ [59]. $G_2/SU(3)$ coset theory does not however reduce to this model as the comparison of conformal weights demonstrates (see Appendix A). From $(c(G_2,14) = 4, c(G_2,7) = 11/6)$ and $c(SU(3),8) = 3, c(SU(3),3) = 4/3)$ the spectrum of conformal weights is $(0, 11/30, 1/30, 1/30)$ for $k(SU(3)) = 1$. Note that the difference between singlet and triplet conformal weights is 1/3.

The beauty of this approach is that QCD might be replaced with an exactly solvable conformal field theory allowing also to deduce how correlation functions change in hyper-octonion analytic transformations affecting space-time surface. There are however also objections against this picture.

1. The basic objection is that G_2 Kac-Moody algebra contains triplet and anti-triplet generators and triplet generators commute to anti-triplet. It is hard to imagine any sensible physical interpretation for these lepto-quark generators, whose commutation relations break the conservation of lepton and quark number.

   The point is however that triplet generators affect e_1, and thus S^6 coordinates and also the $SU(3)$ subgroup acting as isotropy group changes. Thus correlation functions involving these currents are not physically meaningful. Indeed, in G/H coset theory only the H Kac-Moody currents appear naturally in correlation functions since the construction involves functional integral only over H connections [56].

2. If 14-dimensional adjoint representation would appear as primary field, also 3 and $\overline{3}$ lepto-quark like states for which baryon and lepton number are not conserved would appear in the spectrum. The choice $k=1$ provides however a unique manner to circumvent this difficulty. Integrability condition for the highest weight representations allows for a given value of k only the highest weights λ_R satisfying $Tr(\phi\lambda_R) \leq k$, where ϕ is the highest root for Lie-algebra. Since the highest root has length squared 2, adjoint representation is not possible as highest weight representation for $k=1$ WZW model, and the primary fields of G_2 model are singlet and 7-plet corresponding to the hyper-octonionic spinor field and defining in an obvious manner the primary fields $1+3+\overline{3}$ of $G_2/SU(3)$ coset model. Fusion rules for $1 \oplus 7$ correspond to octonionic multiplication. The absence of G_2 gluons saves from lepto-quark like bosons, and the absence of $SU(3)$ gluons can be interpreted as HO counterpart for the fact that all particles, in particular gluons, can be regarded bound states of fermions and anti-fermions in TGD Universe.

   This picture conforms also with the claims that $3+\overline{3}$ part of G_2 algebra does not allow vertex operator construction whereas $SU(3)$ allows the construction in terms of two free bosonic fields. These fields would naturally correspond to the two X^4 directions transversal to the string orbit defined by 1 and e_1. One could say that strings in X^4 are able to represent color Kac-Moody algebra and that $SU(3)$ is inherent to 4-dimensional space-time.

3. The fourth objection is that conformal field theory correlation functions obeying simple scaling laws are not consistent with the exponentially decreasing correlation functions suggested by color confinement. A resolution of the paradox could be based on the role of classical gravitation. At light-like causal determinants the time-like component g_{tt} of the induced metric vanishes meaning that classical gravitational field is very strong. Hence also the normal component g_{nn} of the induced metric is expected to become very large so that hadron would look like the interior of black hole. A finite X^4 proper time for reaching the outer boundary of the hadronic surface can correspond to a very long M^4 time and the finite M^4 distance from the boundary can mean very long distance along hadronic space-time surface. Hence quarks and gluons can behave as almost free particles when viewed from hadronic space-time sheet but look confined when seen from imbedding space. If the hyper-quaternionic coordinates appearing in the correlation functions correspond to internal coordinate of the space-time surface, the correlation functions when expressed in terms of M^4 coordinates can look confining.

5.4 G_2 is very special

It seems that G_2 is exceptional from the point of view of vertex operator construction and that this could allow to understand how TGD relates to bosonic string models and super-string models.

5.4.1 Does vertex operator construction exist for G_2?

G_2 is the only Lie group for which the ratio of the squares of long and short roots is 3. This makes it unique among Lie groups. The following observations are from the article of David Olive reporting the construction of vertex operators for non-simply laced Kac-Moody algebras [47].

1. For the simply-laced Lie algebras A_r ($su(r+1)$), D_r ($so(2r)$), E_6, E_7, and E_8 level $k=1$ representations involve the tachyon emission vertex of the bosonic string theory, and the vertices for all states of the theory generate a Lorentzian algebra of rank 26 based on the unique self-dual even lattice in 26 Lorentzian dimensions.

2. For non-simply laced algebras of type B_r ($so(2r+1)$), C_r ($sp(r)$), F_4 for which the ratio of long roots to short roots equals to 2, the construction of the vertex operators relates to RNS

fermionic string theory with critical dimension 10. The corresponding string theory involves tachyons with mass squared equal to both -2 and -1 corresponding to the two root lengths. The first tachyon decouples from the physical theory although its Regge recurrences occur. Second tachyon decouples in the space-time super-symmetric version of the theory [47, 48]. The construction for these algebras requires the multiplication of the bosonic operators of conformal weight 1/2 (corresponding to roots of length 1) with "fermionic" operators of conformal weight 1/2. The multiplication rules of these fields are fixed from the Kac-Moody algebra relations and there is an interesting connection with the classical division algebras.

3. G_2 is a completely exceptional Lie group since the ratio of lengths squared for long and short roots equals to 3. The obvious question is whether G_2 corresponds to TGD and its reduction to hyper-octonionic string model while other groups would correspond to bosonic and super-string models.

Although [47] does not give the explicit construction of the G_2 vertex operators (it is excluded "for simplicity"), it is clear from the construction of the vertex operators for the other non-simply laced algebras that the construction, if it exists as claimed, can be performed in two parts. The easy part corresponds to the ordinary simply-laced construction for $SU(3) \subset G_2$ having the same Cartan subalgebra as G_2. The remaining generators of G_2 transform as $3+\overline{3}$ under $SU(3)$ and the commutators are of form $[3,3]=\overline{3}$, $[\overline{3},\overline{3}]=3$ and $[3,\overline{3}]=8$. If the vertex operator representation exists, triplet and anti-triplet fields are obtained by multiplying bosonic operators of conformal weight 1/3 with "fermionic" operators having conformal weight 2/3. It seems the anti-commutators of fermionic operators must obey multiplication table of octonionic units in $3+\overline{3}$ if the representation exists.

In [55] it is however argued that the vertex operator construction does not work for G_2. According to [55], for simply laced Lie algebras the representation of level one vertex operators is possible for all primary fields of WZW model. For non-simply laced Lie algebras B_r and C_r only single primary field multiplet of WZW model allows $k=1$ vertex operator representation, whereas the primary fields of G_2 labelling the highest weight states of WZW model with weights c_λ^2/y (c_λ^2 denotes the value of Casimir operator) and F_4 WZW models do not allow vertex operator representations. Since the primary fields of WZW model correspond to chiral and anti-chiral parts of g and to chiral and anti-chiral Kac- Moody generators, the statement contradicts the claim of [47] that vertex operator construction for G_2 exists. Be as it may, it seems that G_2 WZW model differs radically from that associated with simply laced Lie algebras.

5.4.2 G_2 as the minimal option for topological quantum computation

G_2 is also the minimal choice from the point of view of quantum computation [60, 50, E9].

1. The quantum group parameter q associated with the Kac-Moody representation [E9] is determined as $q = exp(i2\pi/y)$, where $y = k + h$ is the sum of the dual Coxeter number h and Kac-Moody central charge k. Recall that the level of representation is defined as $x = 2k/\psi^2$, where $\psi^2 = 2$ is length of the long roots of Lie algebra of G of Kac-Moody representation. $x = 1$ is the lowest non-trivial value of x.

2. From the point of view of topological quantum computation [60, 50, E9] the smallest value of y allowing braid representation of all possible quantum computations is $y = 5$. In this case the phase associated with quantum group parameter $q = exp(i2\pi/y)$ relates very closely to Golden Mean. $h(SU(2)) = 2$ requires $k = 3$ in order to have $y = 5$. On the other hand, $h(G_2) = 4$ means that the lowest level Kac-Moody representation with $x = k = 1$ gives $y = 5$. Hence G_2 gives rise to a minimal Kac-Moody representation allowing the representative power needed by topological quantum computation. Note that $h = 4$ is true

also for $SO(6) = SU(4)$ with dimension $d = 15$ and $Sp(3)$ with dimension $d = 21$. All these groups have rank $r = 3$ whereas G_2 has rank $r = 2$ and dimension $d = 14$ so that G_2 is the minimal choice from the point of view of topological quantum computation.

What is intriguing that the quantum group phase $exp(i2\pi/5)$ appears in the helical structure of DNA proposed in [E9] to act as a topological quantum computer.

One could see these observations as signals for the facts that super strings are not quite enough, and that 8-D WZW strings with pure gauge G_2 gauge potential coupled to hyper-octonionic spinor fields and coding for the space-time surfaces via Kähler calibrations provides the royal road to a unique TOE.

6 $HO - H$ duality and other dualities

During last half year it has become clear that TGD involves an entire web of dualities. Although the picture is far from being crystal clear as yet, general patterns can be already distinguished from the unavoidable cognitive mist surrounding every new idea. These dualities have direct counterparts in hadron physics and allow a fresh view about the relationship between perturbative and non-perturbative QCD, and in fact predict that QCD is quite not correct theory. Spin crisis of proton is one of the phenomena difficult to understand in QCD and finds an elegant explanation in terms of $HO - H$ duality.

6.1 How do $HO - H$ duality, $HQ - coHQ$ duality and electric magnetic duality relate?

The key question is what $HO - H$ duality and other dualities really mean and how they relate to $HQ - coHQ$ duality and electric magnetic dualities.

1. Does HQ-coHQ pairing define a duality? If HQ-coHQ relation defines a duality, the space-time surface associated with a given 3-surface would be different in the dually related pictures. Or should one adopt a weaker interpretation in which $HQ - coHQ$ duality would be more analogous to a map identifying two coordinate patches of the configuration space as a manifold in the region in which they overlap? Only in this region duality would be exact and there might be also 3-surfaces for which the duality would not hold true and they would correspond to coordinate singularities at CH level. At space-time level vacuum extremals with vanishing Kähler fields and CP_2 type extremals would correspond to these two extremes.

2. Do HO and H pictures provide two completely equivalent descriptions of physics or are also these descriptions analogous to different coordinate patches? Is $HO - H$ duality equivalent with $HQ - coHQ$ duality? One might argue that since hyper-octonionic spinor fields are needed to define both HQ and $coHQ$ 4-surfaces, this cannot be the case. There are however indications that CH description could be more convenient in $coHQ$ picture (partons) and CHO description in HQ picture (string orbits).

3. The possibility of electric-magnetic duality at configuration space level was conjectured already more than decade ago, and was inspired by the observation that configuration space Hamiltonians could be defined in terms of either generalized magnetic or electric fluxes [B2, B3]. This duality could naturally correspond to $HQ - coHQ$ duality. Magnetic fluxes are very natural for the flux Hamiltonians defined by the space-like partonic 2-surfaces in H picture, whereas electric fluxes (actually magnetic fluxes for time-like 2-surfaces) are natural for the Hamiltonians associated with time-like string orbits in HO picture. The Kähler functions associated with Kähler metrics defined by these representations would correspond to Kähler function and its dual.

6.1.1 $HQ - coHQ$ duality at the level of configuration space

There are two options concerning the definition of the dual Kähler function in $HQ-coHQ$ duality.

1. Define Kähler function in terms of Kähler calibration using local maximum of L_K instead of minimum. For this option the absolute value of the contribution from a region having fixed sign of Kähler action would be maximized and the option would maximize contrasts and in this sense would be favored by quantum criticality. The strongest pro is that the value of CP_2 Kähler action appears in the expression for the gravitational coupling strength and p-adic evolution for the Kähler coupling strength. The HQ and $coHQ$ pictures would correspond to dove and hawk view about Universe.

2. Replace Kähler action by its dual defined by the normal projection of the induced Kähler form to the normal space associated with given hyper- quaternionic plane. In this case both HQ and $coHQ$ views would represent dove view about Universe and CP_2 action would disappear from the theory as a fundamental parameter. This suggests that this option is not correct. Despite this the explicit definition of the dual of Kähler action deserves a separate discussion.

The dual of the Kähler action would be obtained by replacing the induced CP_2 Kähler form $J_{\alpha\beta}$ with its projection J^N_{kl} to the normal space of the space-time surface. This means a contraction with the projector P to the normal space

$$J^N = PJP \ , \ \ P = h - \nabla h \cdot \nabla h \ , \tag{58}$$

where h denotes the imbedding space metric and $\nabla h \cdot \nabla h$ denotes the H- tensor defined by the space-time inner products of gradients of H coordinates h^k. More explicitly,

$$\begin{aligned} J^N_{kl} &= P_k^{\ r} P_l^{\ s} J_{rs} \ , \\ P^{kl} &= h^{kl} - g^{\mu\nu}\partial_\mu h^k \partial_\nu h^l \ . \end{aligned} \tag{59}$$

The dual action would be defined by the dual action density

$$\begin{aligned} L &= k_1 J^N_{kl} J^{kl}_N \sqrt{g} \ , \\ k_1 &= \frac{1}{16\pi\hat{\alpha}_K} \end{aligned} \tag{60}$$

integrated over the space-time surface.

For both choices of the dual of Kähler action the dual Kähler coupling $\hat{\alpha}_K$ appears as a free parameter. The identification of $\hat{\alpha}_K$ is not quite obvious. The standard form of electric-magnetic duality corresponding to the replacement $g \rightarrow 1/g$ is not sensible in the recent case. This leaves two options.

1. The simplest option would be $\hat{\alpha}_K = \alpha_K$ so that the different dynamics defined by the dual Kähler action would be responsible for the strong–weak duality. This option is not terribly attractive but cannot be excluded.

2. The identification suggested by p-adic length scale hypothesis and $p \leftrightarrow k$ duality of long and short p-adic length scales would be $\hat{\alpha_K}(k) = \alpha_K(p)$ where $p \simeq 2^k$ labels the space-time sheet at which CP_2 type extremals has suffered topological condensation. This identification would

reduce the value of coupling strength and thus also the importance of quantum fluctuations in HO picture at short length scale limit. This identification would suggests that CP_2 type extremal *resp.* the space-time sheet of size of order Compton length at which it is topologically condensed are optimally described as co-hyper-quaternionic *resp.* hyper-quaternionic surface and that the transition from the description to its dual corresponds to the exchange of these surfaces.

6.1.2 $HO-H$ duality at the level of configuration space

An interesting challenge is to translate $HO-H$ duality to the level of configuration space geometry and spinor structure.

1. In H picture CH Hamiltonians correspond to Hamiltonians of $\delta M^4_+ \times CP_2$ in representations of $SO(3)\times SU(3)$ whereas spin and electro-weak spin correspond to spin degrees of freedom associated with complexified gamma matrices acting as super-generators.

2. In HO picture CH is replaced with what might be called CHO. The guess is that also CHO allows Kähler and symplectic structures. CHO Hamiltonians cannot correspond to Hamiltonians of E^7 (imaginary hyper-octonions) since E^7 has wrong dimension. 7-D light-cone is in turn metrically a 6-sphere. If S^6 does not allow complex structure as Chern's last theorem claims, it does not allow Kahler structure neither. Situation changes if one considers $\delta M^4_+ \times E^4$ metrically equivalent to $S^2\times E^4$, which certainly allows Kähler and symplectic structures. This choice is of course perfectly natural and consistent with the view that number theoretical compactification takes effectively E^4 to CP_2 by attaching to it a 2-sphere at infinity. $SO(3)\times SO(4)$ would assign to Hamiltonians spin and ew quantum numbers. Color quantum numbers would correspond to spin degrees of freedom associated with CHO gamma matrices acting also as super generators. $H-HO$ duality could be also interpreted as a super-symmetry permuting bosonic and fermionic degrees of freedom at the level of configuration space.

The obvious question is what is the counterpart of configuration space Kähler function [B1] in HO picture.

1. If $HO-H$ duality is identified with $HQ-coHQ$ duality the situation reduces to the $HQ-coHQ$ duality. It would however seem that the identification of these dualities is in conflict with the fact that HO spinors are needed to define both HQ and $coHQ$ pictures.

2. If the two dualities are not equivalent, the identification of CH and CHO Kähler functions seems to be the most natural option, at least the simplest that one can imagine.

6.2 String-YM duality in TGD framework

Hyper-octonion spinor fields and corresponding G_2 element correspond to an extremal of WZW action plus Dirac action. If the previous arguments make sense, the extremization of Kähler action is equivalent with that of the stringy action. It is perhaps too much to hope that Kähler action is simply a function of WZW action, or even better, proportional to it. Be as it may, this would guarantee the extremum property of Kähler action on basis of hyper-quaternionicity or its co-property.

What gives rise to optimism is that the equivalence of WZW + Dirac action for HO strings with classical color interaction represented by Kähler action (plus the modified Dirac action at light-like causal determinants) conforms with the duality of string models with certain YM theories proposed in M-theory context.

6.3 $HO - H$ duality and ew-color duality

The e(lectro)w(eak)-color duality associated with $H - HO$ duality reflects the fact that in both pictures the dynamics of single space-time surface can provide only a partial description of quantum dynamics, and that configuration space level is needed in order to code all quantum numbers and all interactions. The situation cries for a more precise formulation for the ew–color duality. The sought for formulation can be expressed as a single concise statement.

In H (HO) picture spin and ew-spin (color) degrees freedom correspond to spin like quantum numbers and color (ew) degrees of freedom to classical conserved charges.

6.3.1 Spin-like quantum numbers and conserved charges in H-picture

In H picture ew quantum numbers and spin are manifestly present whereas color quantum numbers and interactions emerge as spin like quantum numbers only at configuration space level as does also four-momentum via Kac-Moody representations. Classical color and Poincare charges are well defined also in H picture. There is also a non-trivial interaction between color and ew degrees of freedom since color transformations are accompanied by ew rotations in accordance with the fact that $U(2)_{ew}$ can be mapped to a subgroup of $SU(3)$ via the coset construction.

6.3.2 Spin-like quantum numbers and conserved charges in HO-picture

Hyper-octonion HO spinors decompose to representations of color group whereas H spinors decompose to the representations of ew and Lorentz group. Hence for HO picture color is manifestly present as spin degrees of freedom but ew spin and spin are absent.

By ew-color duality at space-time level ew and spin charges should somehow emerge also in HO picture as classical conserved quantities. The first observation is that the automorphism group G_2 corresponds to 2 conserved commuting charges. Translating H picture directly to HO level, this would mean that the classical conserved charges associated with WZW + Dirac action have identification as ew charges. Also now a non-trivial relation between electro-weak and color quantum numbers is involved.

There are also symmetries not respecting hyper-octonion real-analyticity and analogous to those affecting the moduli characterizing complex structure. $SO(7)$ leaves the spatial part of the hyper-octonionic norm invariant and this extends the number of conserved charges to 3 bringing in spin. The full isometry group of the hyper-octonionic norm is $SO(7,1)$ so that also Lorentz boost would be included to the Cartan algebra.

Also translations are symmetries of HO picture since the shift of the origin gives rise to a new solution family which is however not hyper-octonion analytic in the original coordinate system. Four- momentum should emerge at quantum level via Kac-Moody type realization also now.

The correspondences $M^4 \times SO(7,1) \leftrightarrow P \times SU(3)$ and $SU(3) \leftrightarrow U(2)_{ew}$ code for the ew–color duality. Interestingly, the four M^4 coordinates depending X^4 coordinates define a local Kac-Moody algebra identifiable in terms of the Cartan algebra of $SO(7,1)$ and extendable by $k = 1$ vertex operator construction to a representation of $SO(7,1)$ Kac-Moody algebra. On the other hand, the Euclidian stringy degrees of freedom in M^4 give rise to $SU(3)$ Kac Moody algebra and to $SU(3)/U(2)$ WZW model serving as a candidate for a model of ew interactions. A very tight web of correspondences between various symmetries is involved.

6.3.3 HO and H pictures: summary

To sum up, the general picture is following.

1. $HO - H$ duality corresponds to two kinds of conformal symmetries: hyper-octonionic conformal invariance and the conformal invariance associated with the light-like causal determi-

nants. Both partonic and string like descriptions are possible and correspond to $HQ - coHQ$ duality.

2. HO picture corresponds to $G_2/SU(3)$ coset WZW theory and naturally to QCD. Color is spin like quantum number and spin and ew spin are orbital quantum numbers at configuration space level.

3. H picture is analogous to $SU(3)/U(2)$ coset theory. H spinor fields are the counterparts of hyper-octonionic spinor fields. The identification of space-time as surface in $M^4 \times CP_2$ is the counterpart of description in terms of $SU(3)/U(2)$ WZW model. In this picture spin and ew spin are spin like quantum numbers and color is at configuration space level. Both partonic and stringy descriptions are possible.

6.4 $HQ - coHQ$-duality, parton-string duality, and generalized Uncertainty Principle

$HQ - coHQ$ duality relates descriptions based on partonic and string like 2-surfaces, and since $HO - H$ duality is different duality, there are four different combinations of these pictures. HQ picture is natural at long length scales when space-time looks like a deformed piece of M^4, whereas $coHQ$ picture is natural at short length scales when CP_2 type extremals dominate. At short (long) length scales $HQ - HO$ ($coHQ - H$) combination looks most natural since fluctuations in configuration space degrees of freedom are minimized and field theory is expected to give reasonable description.

Hyper-quaternionic 4-surfaces can contain real unit and the preferred imaginary hyper-octonionic unit at each point and thus also the string orbit but this is not true in the co-hyper-quaternionic case. Thus string orbits and partonic 2-surfaces belong to dual 4-surfaces intersecting each other at a discrete set of points in the generic case. Obviously a generalization of Uncertainty Principle generalizing ordinary q-p duality to $HQ - coHQ$ duality holds true and HQ and $coHQ$ descriptions relate like function and its Fourier transform. In fact, in the ideal case of foliation each point of a $coHQ$ four-surface corresponds to a HQ four-surface and vice versa, and a kind of Fourier transform assigning to a HQ 4-surface quantum superposition of $coHQ$ surfaces (and vice versa) can be defined.

6.5 Ew-color duality, duality of long and short p-adic length scales, and $(HO, coHQ) - (H, HQ)$ duality

The first formulation [F5] for ew-color duality was in terms of p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, k positive integer, preferably prime or power of prime, as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously this duality would naturally correspond to $HQ - coHQ$ duality.

The proposal was that QCD type description based on quarks and gluons corresponds to a description in the ultra-short length scale L_k and the description in terms of hadrons possessing only electro-weak quantum numbers and spin corresponds to the hadronic length scale L_p. The order of magnitude for α_s is predicted correctly directly from the fact that it is proportional to α_K and as $U(1)$ coupling increases towards short p-adic length scales in a manner predicted by heuristic arguments assuming that gravitational constant does not run appreciably as a function of p-adic length scale.

The duality of p-adic length scales can be interpreted in terms of $HQ-coHQ$ duality. Combining it with electro-weak-color duality, $(HO, coHQ)$ and (H, HQ) pictures emerge as dual pictures. $HO-coHQ$ picture describing color as a spin like quantum number is more appropriate near CP_2 length scale whereas $H-HQ$ picture describing color classically (as in color flux tube models) is more appropriate in hadronic length scales. Perturbative–non-perturbative QCD duality would thus correspond to $(HO, coHQ)-(H, HQ)$ duality. Strictly speaking, non-perturbative QCD in standard sense is of course a meaningless notion in TGD framework.

These arguments led first to the identification of the $HO-H$ and $HQ-coHQ$ dualities. Although it seems practical to use $(HO, coHQ)$ and (H, HQ) pictures implying 1-1 correlation between the dualities, the two dualities are definitely not equivalent.

6.6 Color confinement and its dual as limits when configuration space degrees of freedom begin to dominate

The description of duality at the configuration space level can be applied to gain a view about color confinement and its dual for electro-weak interactions at short distance limit. The correct prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics.

There are two basic types of vacuum extremals: CP_2 type extremals representing elementary particles and vacuum extremals having CP_2 projection which is at most 2-dimensional Lagrange manifold and representing say hadron. It is not surprising that HO-H duality can be interpreted in terms of these vacuum extremals and they provide a more precise view about what happens at the limits when either CH or CHO degrees of freedom begin to dominate over space-time degrees of freedom describable ordinary quantum field theory.

6.6.1 Short distance limit

Consider first the short distance limit at which electro-weak confinement is expected and HO picture becomes more appropriate.

1. Ew-color duality would suggests that at the limit of short distances something analogous to color confinement occurs for electro-weak interactions. Also the large value of $U(1)$ coupling supports this expectation. The vacuum property of CP_2 type extremals means that induced spinor fields become vacuum spinor fields with identically vanishing Dirac action. Therefore these spinor fields effectively disappear at space-time level for the maxima of Kähler function, and contribute only via quantum fluctuations, which correspond to configuration space dynamics. Color partial waves are left as a genuine configuration space degree of freedom and the expectation is that only the lowest color partial waves corresponding to singlet and triplet remain and become spin like degrees of freedom analogous to QCD color in HO picture.

2. Duality suggests $SO(4)$ confinement in E^4 degrees of freedom at this limit. The nearly vacuum property should allow very large fluctuations of the ordinary fermion and anti-fermion numbers at the limit when the fermions become pure vacuons for which creation and annihilation operators reduce to anti-commuting Grassmann numbers. In HO picture this would mean that high $SO(4)$ partial waves in E^4 are possible for composites although net ew quantum numbers vanish. Hence electro-weak spins become analogous to classical angular momentum at this limit.

6.6.2 Long distance limit

Consider next color confinement at the long length scale limit as a dual of this picture.

1. In the case of color interactions very high color partial waves for quarks and gluons appear at the confinement limit. For instance, vacuum extremals representable as maps $M^4 \rightarrow CP_2$ identifiable as hadronic space-time sheets correspond to color confinement limit. Strong fluctuations due to high color partial waves in CH appear, and correspond in CHO picture to the presence of very high colored hyper-octonionic fermion and anti-fermion numbers. Since configuration space degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD just as high energy limit transcends the descriptive power of standard model of electro-weak interactions.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons could directly relate to the fact that this group labels also the E^4 Hamiltonians in HO picture. $SO(4)$ quantum numbers can be identified as right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks.

3. Pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted. The map of electro-weak spin like degrees of freedom to E^4 degrees of freedom maps quark-anti-quark pairs to E^4 coordinates.

4. Baryons should be analogous to color partial waves of quarks, and just as CP_2 spinors allow at CH level color triplet partial waves also hyper-octonionic fermions should allow at CHO level $SO(4)$ partial waves transforming as doublets under $SU(2)_L$ or $SU(2)_R$.

5. Family replication phenomenon is described in the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [F4].

6. Ordinary fermion numbers do not fluctuate at the color confinement limit. That this does not occur must relate to the facts that modified Dirac action relates by super-symmetry to Kähler action and the variations of Kähler action vanish up to third order around canonically embedded M^4 whereas for the CP_2 type extremals the situation is completely different. The absence of fluctuations in ew spin degrees of freedom suggests the possibility of describing low energy hadrons using simple valence quark model without quark color with quark and gluon sea modelling the presence hyper-octonionic quark pairs. The problems due to statistics might be resolved by anyonic statistic possible for 2-D partonic surfaces. At the asymptotic freedom limit hyper-octonionic sea becomes less and less important.

6.6.3 Proton spin crisis as a signature of hyper-octonionic colored quarks?

Hyper-octonionic quarks carry neither ordinary nor electro-weak spin since these quantum numbers correspond to orbital quantum numbers in HO. Hence in the ideal colored quark description the contribution of quarks to both spin and electro-weak spin of proton should vanish whereas in H quark description quarks should give proton spin. Obviously, these descriptions correspond to colored current quark description and to a static color singlet quark descriptions possible for anyonic statistics [61, E9]. This prediction would sound crazy unless the essence of proton spin crisis were just the finding that the contribution of quarks to proton spin is small [62, 63, 64, 65].

Spin-statistics paradox is avoided if configuration space degrees of freedom are taken into account. Quantum-classical correspondence, if taken at extreme, would suggest that configuration space degrees of freedom might have some kind of space-time correlate. The 2-dimensionality of stringy and partonic surfaces suggests that anyons might provide this correlate. In HO picture spin-statistics paradox at space-time level would be avoided by the 2-dimensionality of partonic surfaces allowing to have braid representations of the rotation group and colored quarks can have half-odd integer valued anyonic spin and electro-weak spin.

A possible physical mechanism transforming H quarks without color spin but with ew- and ordinary spin to HO quarks having only color spin is following. Anyonic and ordinary contributions to ew- and ordinary spin of H quark cancel each other and color spin is generated anyonically. In TGD framework anyons are associated with punctures assignable to the thin flux threads connecting partonic 2-surfaces and these punctures appear always as pairs with the ends of thread carrying opposite anyonic quantum numbers. OH fermions would correspond to fermion plus the second end of the anyon thread.

In H picture the approach to confinement means large fluctuations also in $SO(3)$ degrees of freedom and the emergence of Regge trajectories. In HO picture the angular momentum of hadron would be due the angular momentum of a large number of colored quark pairs.

A mechanism resolving not only proton spin crisis but allowing also to understand non-perturbative aspects of hadron physics will be discussed in [F4, F5]. In this framework also super-canonical degrees of freedom contribute to hadron spin and the average spin of nucleonic quark vanishes due to the non-trivial Glebsh-Gordan coupling between proton spin and super-canonical spin. Unlike the more "philosophical" explanation based on OH-O duality, this mechanism is rather predictive: for instance, hadron masses are understood with accuracy better than one per cent.

6.6.4 Summary

It seems that $HO-H$ duality involves an entire web of dualities suggested by the general structure of TGD. Electric-magnetic duality; duality of hyper-quaternionic and co-hyper-quaternionic 4-surfaces; 7-3 duality stating that either space-like 3-surfaces in the intersections of space-time surface with light-like 7-surfaces $\delta M^4\pm \times CP_2$ or light-like 3-surfaces X_l^3 at which X^4 metric becomes degenerate can be taken as causal determinants, and parton-string duality which string theorist would probably call closed-open string duality, would reduce to the same fundamental duality. These dualities correspond to physical dualities such as ew-color duality described above, $p \simeq 2^k \leftrightarrow k$ duality of long and short p-adic length scales, duality of current quarks and static quarks, and duality of hadron and quark level descriptions.

$HO-H$ duality leads to concrete predictions. Some of them are already verified ($SO(4)$ chiral symmetry of low energy hadron physics and explanation of proton spin crisis). Some of them might be testable (colored quarks should not contribute to right and left electro-weak isospin of hadrons). The most important prediction is that both standard model of electro-weak interactions and QCD are in a precisely defined sense wrong theories. One might hope that this might allow to develop an experimental arrangements proving or disproving this prediction.

Spin-statistics paradox was the crucial observation leading to the introduction of quark color. The 2-dimensionality of partonic and stringy surfaces allowing anyonic statistics, or probably equivalently, the presence of configuration space degrees of freedom, allows to circumvent the spin-statistics paradox otherwise implied by the fact that H spinors do not carry color as spin like quantum number and HO spinors do not carry spin and ew spin.

7 A more precise view about $HO-H$ and $HQ-coHQ$ dualities

There are many open questions related to the proposed dualities and the requirement of overall internal consistency and complete symmetry between H and HO pictures gives hopes of achieving a global view about the situation. The understanding of $HO-H$ dualities in terms of momentum and position representations in the cotangent bundle of configuration space reduces the duality to wave-particle duality in infinite-dimensional context. This picture generalizes also to the case of strings and allows to understand what spontaneous compactification means if the notion of stringy configuration space is introduced.

7.1 CHO metric and spinor structure

Configuration space metric is defined by the flux Hamiltonian basis of the configuration space in one-one correspondence with Hamiltonians of $\delta M^4_\pm \times CP_2$ in case of CH and $\delta M^4_\pm \times E^4$ in case of CHO [B2, B3]. E^4 has a natural Kähler structure and the most natural assumption is that E^4 Kähler form defines symplectic and Kähler structure of CHO. CHO Hamiltonians would be defined by Hamiltonians in $\delta M^4_\pm \times E^4$ coordinates belonging in irreducible representations of $SO(3,1) \times SO(4)$.

Also CHO should have spinor structure with gamma matrices acting as super generators. Hyper-octonionic spinor fields with real Laurent coefficients cannot be used to construct super generators and second quantization is needed. The only consistent interpretation is that hyper-octonionic spinor fields correspond to zero modes just as solutions of modified Dirac equation correspond to zero modes.

$HO-H$ duality requires that also second quantized HO spinor fields have $1+1+3+\overline{3}$ decomposition identifiable naturally in terms leptons, quarks and corresponding anti-fermions. What looks strange is that HO spinor field contains components with both lepton and quark number as well as components with opposite quark/lepton numbers. Conservation laws are however respected since $SU(3)$ does not transform these components to each other. In principle the coefficients of second quantized spinor fields are complex numbers commuting with octonion units.

7.2 Can one interpret $HO-H$ duality and $HQ-coHQ$ duality as generalizations of ordinary q-p duality?

It would be highly desirable to reduce the dualities to familiar notions of mathematical physics so that they could be seen as predictions of TGD rather than hypothesis.

7.2.1 $HO-H$ duality and cotangent bundle of CH

A possible interpretation for $CHO-HO$ duality is in terms of ordinary q-p duality generalized to geometric quantization at the level of the cotangent bundle of the configuration space. The points of the fiber of cotangent bundle would be generalizations of canonical momenta. What would be *new* that these canonical momenta would be representable as 4-surfaces in HO, and obtained by assigning to each point of 4-surface of H a co-tangent vector in the fiber of cotangent bundle of H.

The infinite-dimensional generalization of wave-particle duality would allow to use either the base space CH or CHO defining the fiber of cotangent bundle or more generally, Lagrange manifold of the cotangent bundle. If this interpretation is correct, any Lagrange manifold of cotangent bundle is a priori admissible apart from constraints posed by infinite-dimensional existence. Symmetries of course pose natural constraints to the Lagrange manifolds.

Furthermore, covariant gamma matrices are in one-one correspondence with the cotangent space basis so that the permutation of spin and orbital degrees of freedom is consistent with q-p permutation.

A concrete realization of $HO-H$ duality would be in terms of the conserved currents expressible using the canonical momentum densities associated with the Kähler action. For any action density L the canonical momentum densities $\pi_k = \partial L/\partial_t h^k$, where t is some time coordinate, define counterparts of canonical momenta. As functions of X^4 coordinates $\pi_k(x)$ indeed define a 4-surface in $T(H = M^4 \times CP_2)$. The problem is the lack of general covariance at both space-time and imbedding space level.

The symmetric space structure of CP_2 allows to circumvent the problems at the level of imbedding space level. The replacement of the canonical momentum densities by the conserved currents of M^4 translations and of four color isometries corresponding to the complement of Cartan subalgebra $U(2) \subset SU(3)$ guarantees general coordinate invariance at the level of H.

General coordinate invariance can be achieved also at space-time level. The proportionality of the isometry currents to $\sqrt{g}$ spoils their vector field property: the problem is handled by dropping $\sqrt{g}$ out. The resulting coordinates transform like time components of space-time vector fields: scalars are obtained by introducing a preferred time coordinate t. CH is a union of configuration spaces $CH_{\pm}(m)$ assigned to $H_{\pm} = M^4_{\pm} \times CP_2$ with $M^4_{\pm}$ having its tip at point $m \in M^4$, plus more complex configuration spaces associated with the the unions and intersections of future and past light cones. One can assign to $CH_{\pm}(m)$ a unique preferred time coordinate t as the light-cone proper time a and use it in the definition of canonical momentum density.

This realization is physically natural since for vacuum extremals canonical momentum densities vanish identically and dual space-time surface collapses to a point. Quite generally, the dimension of the dual space-time surface is local and can vary in the range $[0, 4]$: perhaps one might speak of TGD counterparts of branes.

7.2.2 $HQ - coHQ$ duality as a generalization of $q - p$ type duality

$HQ - coHQ$ duality brings in mind the generalization of ordinary $q - p$ duality associated with cotangent bundle to a duality defined by the normal bundle of X^4. This requires that space-time dimension is half of the dimension of imbedding space. $HQ - coHQ$ duality would thus permute base and fiber of the normal bundle. What is new that points in the fiber of the normal bundle would have representation as 4-surfaces of H. This is possible if the normal of 4-surface X^4 is identified as a map assigning to each point of X^4 a vector in normal space mapped to point of H somehow so that these points define a surface of H. If the CP_2 part of the normal vector is interpreted as a vector in the complement of $U(2)$ Lie-algebra and exponentiated and projected back to CP_2 a surface in H results.

7.2.3 What is the physical interpretation of $HQ - coHQ$ Fourier transform?

Fourier transform maps functions in base space of cotangent bundle to functions in the fiber of cotangent bundle and both $HO - H$ and $HQ - coHQ$ dualities should correspond to a generalized Fourier transform.

An interesting question is whether $HQ - coHQ$ Fourier transform might have some deeper interpretation. Consider a given HQ 4-surface X^4 and its dual determined by Bohr quantization mediating $HQ - coHQ$ duality. These surfaces have a discrete set of common points in the generic case and one might wonder whether this set of points could have interpretation as punctures representing states created by completely localized (topological?) quantum fields in X^4. The simplest situation is that there exist for each point of X^4 single $coHQ$ surface going through it and this surface intersects X^4 in single point. The questions are following.

1. Is it possible to interpret elements of second quantized Fourier basis in X^4 in terms of functions in the space of 4-surfaces dual to X^4 in this case?
2. What is the interpretation when multiple intersections are possible: do these represent internal degrees of freedom?
3. Could this interpretation give a geometric meaning for n-point functions of quantum fields?

For the foliation of X^4 by stringy 2-surfaces labelled by their partonic duals intersections appear, and the interpretation of the intersection points as punctures representing completely localized states created by conformal or topological quantum field is rather attractive. Also a connection with the 2+2 intersection form of 4-dimensional cohomology playing a key role in the classification of 4-manifolds is suggestive. If this were the case, the intersection form would allow to assign (topological) particle number to space-time surface (or space-time sheet).

7.3 Further implications of $HO-H$ duality

The improved understanding of $HO-H$ duality leads to additional highly non-trivial conjectures.

7.3.1 Does the same 4-D points set represent both q and $p(q)$?

$HO-H$ duality interpreted in terms of cotangent bundle of configuration space forces to ask whether one should not replace Kähler action for $M^4 \times CP_2$ (4-surfaces representing "positions") with its counterpart for $M^4 \times E^4$ in HO picture (4-surfaces representing "momenta"). This would mean a map assigning to a given 4-surface in H its dual in HO. Kind of Bohr quantization would be in question: to a given position of particle (4-surface in H) a unique momentum (4-surface in HO would be assigned.

The idea that these two different 4-surfaces should to different point sets in HO and H does not seem attractive, and the question is whether the same point-set X^4 in HO represents both CH point and CHO point (position q and momentum $p(q)$) and the differences come only from the fact that the metric and Kähler form are induced from HO and H respectively. This would be of course in spirit with the notion of $HO-H$ duality and provide it with an additional beauty and meaning. Also Lagrange manifolds applied in geometric quantization assign unique momentum to given position and thus perform Bohr orbit quantization.

The duality in this strong sense raises several interesting questions.

1. Do vacuum extremals for HO Kähler action co-incide set-theoretically with those for H Kähler action?
2. Is the value of Kähler function defined by the two Kähler actions same for a given 3-surface?

These two requirements imply that all 4-surfaces $X^4 \subset HO$ approach to vacuum extremals at large distances from the origin and asymptotically have at most 2-D E^4 and CP_2 projections. These surfaces need not of course have infinite size in HO.

Infinite primes have interpretation as Fock states and representation as 4-surfaces [E3]. The hyper-octonionic building bricks of infinite primes allow a straightforward interpretation as components of a quantized 8-momentum for a particle in HO and thus correspond to the momentum representation of the theory.

7.3.2 Do both HO and H spinor fields define foliations?

Complete $HO-H$ duality suggests that hyper-octonionic spinor fields should have analogs at the level of H. Solutions of massless Dirac equation for H-spinors are the only reasonable candidates for the counterparts of H spinor fields. These spinor fields should define in some natural manner a map assigning to a point of H a point of HO. The natural guess is that M^4 coordinates result by canonical projection whereas E^4 coordinates e^k correspond to the currents $e^k = \overline{\Psi}\Gamma^k_{CP_2}\Psi$. Ψ is fixed apart from a local $U(1)$ phase and the consistency condition would be that Ψ (which have either quark or leptonic chirality) satisfies massless Dirac equation in H. If a superposition of quark and lepton like currents is allowed, the changes of satisfying the condition are better. The Kähler calibration for HO Kähler action should define same 4-surface in set theoretic sense as that for H Kähler action.

7.3.3 Are induced spinor fields restrictions of imbedding space spinor fields?

As already found, the strict $HO-H$ duality supports the view that the induction to X^4 makes sense for both H and HO spinor fields, that the zero modes of these spinor fields satisfying massless Dirac equation at the level of H (HO) and modified Dirac equation at the level of $X^4 \subset H$ ($X^4 \subset HO$) correspond to classical degrees of freedom, and that second quantized part defines the gamma matrices of configuration space CH (CHO) acting as super-generators.

The introduction of imbedding space and space-time spinor fields as independent dynamical degrees of freedom does not look a good idea and the simplest assumption is that the allowed induced spinor fields or at least the zero modes for these spinor fields in the foliation correspond to the restrictions of H and HO spinor fields. This would select only single zero mode from the space of all allowed ones.

For the second quantized X^4 spinor fields the representability as a restriction of second quantized H spinor field would mean that the anti-commutators of H spinor fields are non-vanishing at surfaces X^4 rather than at 7-D surfaces as in ordinary quantization. This would resolve the longstanding problem how to perform induction procedure for quantized H spinor fields without producing fatal 7-D delta function singularities to the space-time integrals of anti-commutators of the induced spinor fields.

If the Laurent coefficients of the zero mode spinor fields are Hermitian operators representing observables characterizing quantum states this selection would realize quantum-classical correspondence. Space-time surfaces would serve as representations of quantum states. This assumption is however not absolutely essential and the question of how this correspondence is realized remains unanswered.

7.4 Do induced spinor fields define foliation of space-time surface by 2-surfaces?

$HO-H$ duality in its strongest form would mean a foliation of HO/H by HQ 4-surfaces with the surfaces of the foliation parameterized by the points of any *coHQ* surface. This foliation is defined by HO spinor field.

One can ask whether the zero modes of induced H spinor field in X^4 could define a foliation of 4-surface. For HQ 4-surface regarded as surface in H this foliation would be by hyper-complex (HC) stringy 2-surfaces labelled by the points of any *coHC* partonic surface. For *coHQ* 4-surface both 2-surfaces and co-2-surfaces would be *coC* surfaces.

How this foliation could be defined by a zero mode of induced H or HO spinor field? The following argument translates the construction in HO case to HQ case.

1. The first guess is that the map $HO \rightarrow M^4 \times CP_2$ generalizes to a map $X^4 \rightarrow M^2 \times S^2$. This means that zero mode must assign to a given point of X^4 points of M^2 and S^2. This map in turn would define a foliation of X^4 by HC 2-surfaces labelled points of any *coHC* 2-surface so that string model like structure would result.

2. Point of M^2 could be assigned to a point X^4 by selecting a preferred subspace of M^4 an applying canonical projection from X^4 to M^2. Lorentz invariance is a possible source of troubles.

3. One must consider the situation in both HO and H pictures.

   i) The hyper-octonionic spinor field mode of HO induced to space-time surface would has with respect to the quaternionic automorphism group $SO(3)$ $1+3$ decomposition.

   ii) For a given zero mode of H spinor field with given chirality could be regarded as a 2-component spinor field having two complexified quaternionic components and obeying Weyl condition. If one allows the representation in which quaternionic units are expressed in terms of Pauli sigma matrices, it decomposes into two doublets under the automorphism group $SU(2)$ of complexified quaternions. Also an additional $U(1)$ factor appears naturally from complexification. The identification as representations of electro-weak $U(2)$ would be indeed very natural. If quaternionic units are representable as in terms of induced H gamma matrices this decomposition is natural.

4. Assume a selection of preferred hyper-quaternionic imaginary unit. The automorphisms leaving this unit invariant correspond to the rotation group $SO(2)$ and the possible selections of this unit are labelled by points of S^2 just like in the case of HO the selections remain invariant under $SU(3)$ and are labelled by S^6. S^2 degree of freedom corresponds to a freedom to perform a local rotation for the imaginary part of the argument in hyper-quaternionic power series. That the groups $U(1)$ and $SU(3)$ are exact symmetries of standard model might relate to the foliations in some deep manner. In H picture $U(1) \times U(1)$ defines the invariance group and gives $S^2 = U(2)/U(1) \times U(1)$.

5. In HO picture the tensor product $3 \otimes 3$ for triplet part of Ψ contains triplet identifiable as $SO(3)$ Lie algebra element which can be exponentiated so that it assigns a point of S^2 just as HO spinor field defines point of CP_2. Also the doublet decomposition natural in H picture allows this kind of map. The point of S^2 defines a hyper-complex plane at each point of X^4. Integrable distribution of these planes possibly defined by the analog of Kähler calibration would define a foliation of X^4 by 2-surfaces and its dual.

6. WZW action would presumably emerge as the counterpart of Kähler action now and would define $SO(3)/SO(2)$ coset model in HO picture. In H picture $SO(3))$ is replaced by $U(2)$ and $U(2)/U(1) \times U(1)$ coset model results. These models must be equivalent. $U(2)$ would be identifiable as electro-weak gauge group so that this step in reduction would correspond to electro-weak symmetry breaking. Since quaternions are complexified the $U(1)$ factor emerges naturally.

There are obvious generalizations.

1. The construction should generalize also to *coHQ* surfaces. In this case, the dual 2-surfaces would be naturally complex both.

2. Similar foliation by 2-surfaces and their co-2-surfaces should appear also in HO picture and for the induced spinors obtained from those of HO satisfying the modified Dirac equation defined for the counterpart of Kähler action in HO. A further question inspired by Bohr quantization and duality considerations is whether these foliations are identical set-theoretically and the differences come only from the induced metric and Kähler structure.

7.5 Web of coset theories?

It seems that there is a web of WZW type models. $G_2/SU(3)$ would correspond to QCD type theory, $SU(3)/U(2)$ to electro-weak theory, $U(2)/U(1)$ to QED type theory. Without having a deeper knowledge in arrowlogy I cannot avoid the temptation of pondering whether the following commutative diagram might have some deep significance.

$$
\begin{array}{ccccc}
U(1) & \rightarrow & U(2) & \longrightarrow & U(2)/U(1) = S^2 \\
\downarrow & & \downarrow & & \uparrow \\
U(2) & \rightarrow & SU(3) & \longrightarrow & SU(3)/U(2) = CP_2 \\
\downarrow & & \downarrow & & \uparrow \\
SU(3) & \rightarrow & G_2 & \longrightarrow & G_2/SU(3) = S^6 \\
\downarrow & & \downarrow & & \uparrow \\
SO(7) & \rightarrow & SO(7,1) & \longrightarrow & SO(7,1)/SO(7) = H_7
\end{array}
\quad (61)
$$

7.6 Could configuration space cotangent bundle allow to understand M-theory dualities at a deeper level?

It is interesting to see* how TGD dualities relate to U, T and S dualities of M-theory. T duality relates large and small scales, S duality $g \leftrightarrow 1/g$ relates coupling constants and U dualities correspond to products of S and T dualities. Obviously $HQ - coHQ$ duality acts like T that it relates long and short length scales (M^4 type and CP_2 vacuum extremals in extreme situation). In the case of $HQ - coHQ$ duality the Z_2 symmetry permutes minimum and maximum of L_K and tangential and normal degrees of freedom. In the case of $HO - H$ duality spin and orbital degrees of freedom are permuted by Z^2 symmetry. In cotangent bundle picture fiber and base degrees of freedom are exchanged. If the coupling constants associated with k and $p \simeq 2^k$ are mapped to each other, also S duality aspect is involved.

$HO - H$ duality and the generalization of configuration space of 3-surfaces to that of configuration space of 1-surfaces with Kähler metric (which need not exist mathematically except in very rare special cases) allows also a deeper understanding of spontaneous compactification and the dualities of M-theory.

Stringy compactification could be understood in terms of the cotangent bundle of stringy configuration space of strings and its dual defining the representations of canonical momenta and positions in the configuration spaces of strings associated with non-compactified and compactified target space respectively. This would explain why a given flat space theory allows several compactifications. The non-compactified theory would correspond to the perturbative theory which is always the same whereas compactifications would correspond to non-perturbative theories, which are the "real" theories. The duality mapping strings in two representations to each other would be analogous to Bohr quantization assigning to a given position q canonical momentum p(q), and Lagrange manifolds of cotangent bundle of compactified theory would determine a huge variety of different dualities unless there are some constraints from symmetries. Probably these constraints are very important, at least in TGD.

The mysterious dualities would thus reduce to a rather familiar notion of cotangent bundle, the representation of points of the fiber of this bundle as 2-surfaces, and the realization of Bohr orbitology using a generalization of Lagrange manifold allowing to map points of base-space to those of fiber.

Of course, this formal picture need not make sense since the existence of infinite-dimensional

Kähler geometry poses so strong constraints that the notion of dynamical target space must be given up since for an arbitrary target space configuration space geometry need not exist at all. This would of course mean getting rid of the landscape problem. Unfortunately, it could also mean that the physics predicted by string models and M-theory does not have much to do with what we see in laboratory.

For instance, the Kähler geometry of loop groups is unique and has Kac-Moody algebras as isometry algebras. This would suggest that the most general choice for target space is as a product of loop groups containing also Abelian factors. Perhaps also loop analogs of products of coset spaces are possible. 2-dimensional general coordinate invariance gives strong constraints and Wess-Zumino-Witten action seems to be a natural candidate for the bosonic part of string action. TGD suggests that strings in 4-D space is the only possible option since it allows the analog of HQ-coHQ duality as an additional symmetry. This would however trivialize the theory since internal symmetries would be absent.

7.7 E_8 theory of Garrett Lisi and TGD

Recently (towards end of the year 2007) there has been a lot of fuss about the E_8 theory proposed by Garrett Lisi [51] in physics blogs, in media, and even New Scientist [52] wrote about the topic. There are serious objections against Lisi's theory and it is interesting to find whether the theory could be modified so that it would survive the basic objections. Although it seems that Lisi's theory cannot be saved, one achieves further insights about HO-H duality. Number theoretical spontaneous compactification can be formulated in terms of the Kac-Moody algebra assignable to Poincare group and standard model gauge group having also rank 8. The representation can be constructed in standard manner using quantized M^8 coordinates at partonic 2-surfaces. Also E_8 representations are in principle possible and the question concerns their physical interpretation.

7.7.1 Objections against Lisis theory

The basic claim of Lisi is that one can understand the particle spectrum of standard model in terms of the adjoint representation of a noncompact version E_8 group [53]. There are several objections against E_8 gauge theory interpretation of Lisi.

1. Statistics does not allow to put fermions and bosons in the same gauge multiplet. Also the identification of graviton as a part of a gauge multiplet seems very strange if not wrong since there are no roots corresponding to a spin 2 two state.

2. Gauge couplings come out wrong for fermions and one must replace YM action with an ad hoc action.

3. Poincare invariance is a problem. There is no clear relationship with the space-time geometry so that the interpretation of spin as E_8 quantum numbers is not really justified.

4. Finite-dimensional representations of non-compact E_8 are non-unitary. Non-compact gauge groups are however not possible since one would need unitary infinite-dimensional representations which would change the physical interpretation completely. Note that also Lorentz group has only infinite-D unitary representations and only the extension to Poincare group allows to have fields transforming according to finite-D representations.

5. The prediction of three fermion families is nice but one can question the whole idea of putting particles with mass scales differing by a factor of order 10^{12} (top and neutrinos) into same multiplet. For some reason colleagues stubbornly continue to see fundamental gauge symmetries where there seems to be no such symmetry. Accepting the existence of

a hierarchy of mass scales seems to be impossible for a theoretical physicist in main main stream although fractals have been here for decades.

6. Also some exotic particles not present in standard model are predicted: these carry weak hyper charge and color (6-plet representation) and are arranged in three families.

7.7.2 Three attempts to save Lisis theory

To my opinion, the shortcomings of E_8 theory as a gauge theory are fatal but the possibility to put gauge bosons and fermions of the standard model to E_8 multiplets is intriguing and motivatse the question whether the model could be somehow saved by replacing gauge theory with a theory based on extended fundamental objects possessing conformal invariance.

1. In TGD framework H-HO duality allows to consider Super-Kac Moody algebra with rank 8 with Cartan algebra assigned with the quantized coordinates of partonic 2-surface in 8-D Minkowski space M^8 (identifiable as hyper-octonions HO). The standard construction for the representations of simply laced Kac-Moody algebras allows quite a number of possibilities concerning the choice of Kac-Moody algebra and the non-compact E_8 would be the maximal choice.

2. The first attempt to rescue the situation would be the identification of the weird spin 1/2 bosons in terms of supersymmetry involving addition of righthanded neutrino to the state giving it spin 1. This options does not seem to work.

3. The construction of representations of non-simply laced Kac-Moody algebras (performed by Goddard and Olive at eighties [47]) leads naturally to the introduction of fermionic fields for algebras of type B, C, and F: I do not know whether the construction has been made for G_2. E_6, E_7, and E_8 are however simply laced Lie groups with single root length 2 so that one does not obtain fermions in this manner.

4. The third resuscitation attempt is based on fractional statistics. Since the partonic 2-surfaces are 2-dimensional and because one has a hierarchy of Planck constants, one can have also fractional statistics. Spin 1/2 gauge bosons could perhaps be interpreted as anyonic gauge bosons meaning that particle exchange as permutation is replaced with braiding homotopy. If so, E_8 would not describe standard model particles and the possibility of states transforming according to its representations would reflect the ability of TGD to emulate any gauge or Kac-Moody symmetry.

The standard construction for simply laced Kac-Moody algebras might be generalized considerably to allow also more general algebras and fractionization of spin and other quantum numbers would suggest fractionization of roots. In stringy picture the symmetry group would be reduced considerably since longitudinal degrees of freedom (time and one spatial direction) are non-physical. This would suggest a symmetry breaking to $SO(1,1)\times E_6$ representations with ground states created by tachyonic Lie allebra generators and carrying mass squared 2 in suitable units. In TGD framework the tachyonic conformal weight can be compensated by super-canonical conformal weight so that massless states getting their masses via Higgs mechanism and p-adic thermodynamics would be obtained.

7.7.3 Could super-symmetry rescue the situation?

E_8 is unique among Lie algebras in that its adjoint rather than fundamental representation has the smallest dimension. One can decompose the 240 roots of E_8 to 112 roots for which two components of SO(7,1) root vector are $\pm$ 1 and to 128 vectors for which all components are $\pm$ 1/2 such that the sum of components is even. The latter roots Lisi assigns to fermionic states. This is not consistent with spin and statistics although SO(3,1) spin is half-integer in M^8 picture.

The first idea which comes in mind is that these states correspond to super-partners of the ordinary fermions. In TGD framework they might be obtained by just adding covariantly constant right-handed neutrino or antineutrino state to a given particle state. The simplest option is that fermionic super-partners are complex scalar fields and sbosons are spin 1/2 fermions. It however seems that the super-conformal symmetries associated with the right-handed neutrino are strictly local in the sense that global super-generators vanish. This would mean that super-conformal super-symmetries change the color and angular momentum quantum numbers of states. This is a pity if indeed true since super-symmetry could be broken by different p-adic mass scale for super partners so that no explicit breaking would be needed.

7.7.4 Could Kac Moody variant of E_8 make sense in TGD?

One can leave gauge theory framework and consider stringy picture and its generalization in TGD framework obtained by replacing string orbits with 3-D light-like surfaces allowing a generalization of conformal symmetries.

H-HO duality is one of the speculative aspects of TGD. The duality states that one can either regard imbedding space as $H = M^4 \times CP_2$ or as 8-D Minkowski space M^8 identifiable as the space HO of hyper-octonions which is a subspace of complexified octonions. Spontaneous compactification for M^8 described as a phenomenon occurring at the level of Kac-Moody algebra would relate HO-picture to H-picture which is definitely the fundamental picture. For instance, standard model symmetries have purely number theoretic meaning in the resulting picture.

The question is whether the non-compact E_8 could be replaced with the corresponding Kac Moody algebra and act as a stringy symmetry. Note that this would be by no means anything new. The Kac-Moody analogs of E_{10} and E_{11} algebras appear in M-theory speculations. Very little is known about these algebras. Already $E < sub > n < /sub >$, $n > 8$ is infinite-dimensional as an analog of Lie algebra. The following argument shows that E_8 representations do not work in TGD context unless one allows anyonic statistics.

1. In TGD framework space-time dimension is D=8. The speculative hypothesis of HO-H duality inspired by string model dualities states that the descriptions based on the two choices of imbedding space are dual. One can start from 8-D Cartan algebra defined by quantized M^8 coordinates regarded as fields at string orbit just as in string model. A natural constraint is that the symmetries act as isometries or holonomies of the effectively compactified M^8. The article "The Octonions" [29] of John Baez discusses exceptional Lie groups and shows that compact form of E_8 appears as isometry group of 16-dimensional octo-octonionic projective plane $E_8/(Spin(16)/Z_2)$: the analog of CP_2 for complexified octonions. There is no 8-D space allowing E_8 as an isometry group. Only SO(1,7) can be realized as the maximal Lorentz group with 8-D translational invariance.

2. In HO picture some Kac Moody algebra with rank 8 acting on quantized M^8 coordinates defining stringy fields is natural. The charged generators of this algebra are constructible using the standard recipe involving operators creating coherent states and their conjugates obtained as operator counterparts of plane waves with momenta replaced by roots of the simply laced algebra in question and by normal ordering.

3. Poincare group has 4-D maximal Cartan algebra and this means that only 4 Euclidian dimensions remain. Lorentz generators can be constructed in standard manner in terms of Kac-Moody generators as Noether currents.

4. The natural Kac-Moody counterpart for spontaneous compactification to CP_2 would be that these dimensions give rise to the generators of electro-weak gauge group identifiable as a

product of isometry and holonomy groups of CP_2 in the dual H-picture based on $M^4 \times CP_2$. Note that in this picture electro-weak symmetries would act geometrically in E^4 whereas in CP_2 picture they would act only as holonomies.

Could one weaken the assumption that Kac-Moody generators act as symmetries and that spin-statistics relation would be satisfied?

1. The hierarchy of Planck constants relying on the generalization of the notion of imbedding space breaks Poincare symmetry to Lorentz symmetry for a given sector of the world of classical worlds for which one considers light-like 3-surfaces inside future and past directed light cones. Translational invariance is obtained from the wave function for the position of the tip of the light cone in M^4. In this kind of situation one could consider even E_8 symmetry as a dynamical symmetry.

2. The hierarchy of Planck constants involves a hierarchy of groups and fractional statistics at the partonic 2-surface with rotations interpreted as braiding homotopies. The fractionization of spin allows anyonic statistics and could allow bosons with anyonic half-odd integer spin. Also more general fractional spins are possible so that one can consider also more general algebras than Kac-Moody algebras by allowing roots to have more general values. Quantum versions of Kac-Moody algebras would be in question. This picture would be consistent with the view that TGD can emulate any gauge algebra with 8-D Cartan algebra and Kac-Moody algebra dynamically. This vision was originally inspired by the study of the inclusions of hyper-finite factors of type II¡sub¿1¡/sub¿. Even higher dimensional Kac-Moody algebras are predicted to be possible.

3. It must be emphasized that these considerations relate in TGD framework to Super-Kac Moody algebra only. The so called super-canonical algebra is the second quintessential part of the story. In particular, color is not spin-like quantum number for quarks and quark color corresponds to color partial waves in the world of classical worlds or more concretely, to the rotational degrees of freedom in CP_2 analogous to ordinary rotational degrees of freedom of rigid body. Arbitrarily high color partial waves are possible and also leptons can move in triality zero color partial waves and there is a considerable experimental evidence for color octet excitations of electron and muon but put under the rug.

7.7.5 Can one interpret three fermion families in terms of E_8 in TGD framework?

The prediction of three fermion generations by E_8 picture must be taken very seriously. In TGD three fermion generations correspond to three lowest genera $g = 0, 1, 2$ (handle number) for which all 2-surfaces have Z_2 as global conformal symmetry (hyper-ellipticity [F1, F2]). One can assign to the three genera a dynamical SU(3) symmetry. Theye are related by SU(3) triality which brings in mind the triality symmetry acting on fermion generations in E_8 model. SU(3) octet and singlet bosons correspond to pairs of light-like 3-surfaces defining the throats of a wormhole contact and since their genera can be different one has color singlet and octet bosons. Singlet corresponds to ordinary bosons. Color octet bosons must be heavy since they define neutral currents between fermion families.

The three E_8 anyonic boson families cannot represent family replication since these symmetries are not local conformal symmetries: it obviously does not make sense to assign a handle number to a given point of partonic 2-surface! Also bosonic octet would be missing in E_8 picture.

One could of course say that in E_8 picture based on fractional statistics, anyonic gauge bosons can mimic the dynamical symmetry associated with the family replication. This is in spirit with the idea that TGD Universe is able to emulate practically any gauge - or Kac-Moody symmetry and that TGD Universe is busily mimicking also itself.

To sum up, the rank 8 Kac-Moody algebra - emerging naturally if one takes HO-H duality seriously - corresponds very naturally to Kac-Moody representations in terms of free stringy fields for Poincare-, color-, and electro-weak symmetries. One can however consider the possibility of anyonic symmetries and the emergence of non-compact version of E_8 as a dynamical symmetry, and TGD suggests much more general dynamical symmetries if TGD Universe is able to act as the physics analog of the Universal Turing machine.

8 Appendix A: Is $G_2/SU(3)$ coset model a rational conformal field theory?

$G_2/SU(3)$ coset model model has central charge $c = 3/4$, which corresponds to rational conformal field theory. The question is whether $G_2/SU(3)$ model could reduce to $c = 3/4$ rational CFT.

Let us recall the standard facts about rational conformal field theories. The reason for their nice properties is that there exists an infinite number of conformal fields, which create degenerate states having zero norm as states of Virasoro algebra.

1. For a given value c of the Virasoro central charge, the degenerate states appear for special values of the conformal weights given by Kac formula

$$\begin{array}{ll} \Delta_{m,n} = \Delta_0 + \frac{1}{4}(\alpha_+ m + \alpha_- n)^2 \ , & \\ \Delta_0 = \frac{1}{24}(c-1) \ , & \alpha_\pm = \frac{\sqrt{1-c}\pm\sqrt{25-c}}{\sqrt{24}} \ . \end{array} \quad (62)$$

   The requirement that conformal weights are real and positive excludes $c > 1$. $c < 1$ is however not enough to guarantee positivity of conformal weights.

2. The primary fields labelled by the integers m and n satisfy the fusion rules

$$\Psi_{m_1,n_1}\Psi_{m_2,n_2} = \sum_{k=|m_1-m_2|+1}^{m_1+m_2+1} \sum_{l=|n_1-n_2|+1}^{n_1+n_2+1} \Psi_{k,l} \ . \quad (63)$$

   The fusion algebra is infinite unless the number of degenerate states is infinite.

3. If the ratio α_+/α_- is irrational, the value of $\Delta_{m,n}$ can be arbitrary near to Δ_0 and thus negative, which is physically unacceptable. Thus physical considerations force the rationality assumption

$$-\frac{\alpha_-}{\alpha_+} = \frac{p}{q} \ , \quad (64)$$

   where p and q are positive integers. The numbers of primary fields in this case is finite since $0 < m < p$, $0 < n < q$ holds true.

From $c = 3/4$ the values of integers p and q are $p = 5$, $q = 6$ so that the number of primary fields $\Psi_{m,n}$, $0 < m < p$, $0 < n < 1$ of the coset model would be $(p-1)(q-1) = 20$. There are only 10 different pairs conformal weights. The list of the weights is given by the following table

n/m	1	2	3	4
5	3	7/5	2/5	0
4	13/8	21/40	1/40	1/8
3	2/3	1/15	1/15	2/3
2	1/8	1/40	21/40	13/8
1	0	2/5	7/5	3

Table 1. The conformal weights of 20 primary fields of $c = 3/4$ rational QFT.

The odd rows of the table give the conformal weights of the three state Potts model (in the fusion algebra the products of odd rows give only odd rows). $\Delta = 3$ corresponds to the conformal weight of the W field of W_3 algebra realized as a third order $SU(3)$ Casimir operator trilinear and completely symmetric in $SU(3)$ Kac-Moody generators [59].

Three-state Potts model has been proposed as a model for the critical behavior of 2+1-dimensional QCD in confinement-de-confinement transition [58], and it is interesting to see whether $G_2/SU(3)$ theory could reduce to it. The conformal weights of $G_2/SU(3)$ theory are given by the differences $\Delta = c(G_2, R)/5 - c(SU(3), R')/4$, where R corresponds to 1-D and 7-D representations of G_2 and R' to the representations $1, 1, 3, \overline{3}$ of SU(3). It is obvious that the primary fields corresponding to $1, 1, 1, 3$, and $\overline{3}$ cannot correspond to the primary of the rational CFT. Neither the resulting weight spectrum $0, c(G_2, 7)/5, c(G_2, 7)/5 - 2/3$ can correspond to the weights appearing in the table. Also the upper bound $\Delta \leq c(G_2, 7) < c(G_2, 14) = 4/5$ fails to be satisfied. The spectrum of conformal weights is $(0, 11/30, 1/30, 1/30)$.

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