Physics as a Generalized Number Theory

M. Pitkänen[1], February 1, 2006

[1] Department of Physical Sciences, High Energy Physics Division,
PL 64, FIN-00014, University of Helsinki, Finland.
matpitka@rock.helsinki.fi, http://www.physics.helsinki.fi/∼matpitka/.
Recent address: Puutarhurinkatu 10,10960, Hanko, Finland.

Contents

1 Introduction

2 TGD as a Generalized Number Theory: p-Adicization Program

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation. The original chapter representing the number theoretic vision as a consistent narrative grew so massive that I decided to divide it to three parts.

The first part is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

Second part focuses on the idea that the tangent spaces of space-time and imbedding space can be regarded as 4- *resp.* 8-dimensional algebras such that space-time tangent space defines sub-algebra of imbedding space. The basic candidates for the pair of algebras are hyper-quaternions and hyper-octonions.

The great idea is that space-time surfaces X^4 correspond to hyper-quaternionic or co-hyper-quaternionic sub-manifolds of $HO = M^8$. The possibility to assign to X^4 a surface in $M^4 \times CP_2$ means a number theoretic analog for spontaneous compactification. Of course, nothing dynamical is involved: a dual relation between totally different descriptions of the physical world are in question. In the spirit of generalized algebraic geometry one can ask whether hyper-quaternionic space-time surfaces and their duals could be somehow assigned to hyper-octonion analytic maps $HO \rightarrow HO$, and there are good arguments suggesting that this is the case.

The third part is devoted to infinite primes. Infinite primes are in one-one correspondence with the states of super-symmetric arithmetic quantum field theories. The infinite-primes associated with hyper-quaternionic and hyper-octonionic numbers are the most natural ones physically because of the underlying Lorentz invariance, and the possibility to interpret them as momenta with mass squared equal to prime. Most importantly, the polynomials associated with hyper-octonionic infinite primes have automatically space-time surfaces as representatives so that space-time geometry becomes a representative for the quantum states.

2.1 The painting is the landscape

The work with TGD inspired theory of consciousness has led to a vision about the relationship of mathematics and physics. Physics is not in this view a model of reality but objective reality itself: painting is the landscape. One can also equate mathematics and physics in a well defined sense and the often implicitly assumed Cartesian theory-world division disappears. Physical realities are mathematical ideas represented by configuration space spinor fields (quantum histories) and quantum jumps between quantum histories give rise to consciousness and to the subjective existence of mathematician.

The concrete realization for the notion algebraic hologram based on the notion of infinite prime is a second new element. The notion of infinite rationals leads to the generalization of also the notion of finite number since infinite-dimensional space of real units obtained from finite rational valued ratios q of infinite integers divided by q. These units are not units in p-adic sense. The generalization to the quaternionic and octonionic context means that ordinary space-time points become infinitely structured and space-time point is able to represent even the quantum physical state of the Universe in its algebraic structure. Single space-time point becomes the Platonia not visible at the level of real physics but essential for mathematical cognition.

In this view evolution becomes also evolution of mathematical structures, which become more and more self-conscious quantum jump by quantum jump. The notion of p-adic evolution is indeed a basic prediction of quantum TGD but even this vision might be generalized by allowing rational-adic topologies for which topology is defined by a ring with unit rather than number field.

2.2 Real and p-adic regions of the space-time as geometric correlates of matter and mind

The solutions of the equations determining space-time surfaces are restricted by the requirement that the components of quaternions are real. When this is not the case, one might apply instead of a real completion with some rational-adic or p-adic completion: this is how rational-adic p-adic physics emerges from basic equations of the theory. One can interpret the resulting rational-adic or p-adic regions as geometrical correlates for 'mind stuff'.

p-Adic non-determinism implies extreme flexibility and therefore makes the identification of the p-adic regions as seats of cognitive representations very natural. Unlike real completion, p-adic completions preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with 'mind like' regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of 'self' and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

2.3 The generalization of the notion of number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

2.4 p-Adicization by algebraic continuation

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function.

For instance, residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition and intentionality. The basic stumbling block of this program is integration and algebraic continuation should allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. Two types of cutoffs are predicted: p-adic length scale cutoff and a cutoff due to phase resolution. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and Beraha numbers $B_n = 4cos^2(\pi/n)$, $n \geq 3$ related closely to the hierarchy of quantum groups, braid groups, and II_1 factors of von Neumann algebra [E10]. This cutoff hierarchy seems to relate closely to the hierarchy of cutoffs defined by the hierarchy of subalgebras of the super-canonical algebra defined by the hierarchy of sets $(z_1, ...z_n)$, where z_i are the first n non-trivial zeros of Riemann Zeta [C5]. Hence there are good hopes that the p-adicization program might unify apparently unrelated branches of mathematics.

3 How p-adic and real physics relate at space-time level?

The interpretation of the p-adic as physics of cognition and the vision about reduction of physics to rational physics continuable algebraically to various extensions of rationals and p-adic number fields is an attractive general framework allowing to understand how p-adic fractality could emerge in real physics. In this section it will be found that this vision provides a concrete tool in principle allowing to construct global solutions of field equations by reducing long length scale real physics to short length scale p-adic physics. Also p-adic length scale hypothesis can be understood and the notion of multi-p p-fractality can be formulated in precise sense in this framework. This vision leads also to a concrete quantum model for how intentions are transformed to actions and the S-matrix for the process has the same general form as the ordinary S-matrix.

The fractal hierarchy associated with Golden mean cannot be understood in a manner analogous to p-adic fractal hierarchies. Rather, the understanding of Golden Mean and Fibonacci series could reduce to the hypothesis that space-time surfaces, and thus the geometry of physical systems, provide a representations for the hierarchy of Fibonacci numbers characterizing the Jones inclusions of infinite-dimensional Clifford sub-algebras of configuration space spinors identifiable as infinite-dimensional von Neumann algebras known as hyper-finite factors of type II_1 (not that configuration space corresponds here to the "world of classical worlds"). The emergence of powers of e has been discussed in [E8] and will not be discussed here.

3.1 p-Adic physics and the construction of solutions of field equations

The number theoretic vision about physics relies on the idea that physics or, rather what we can know about it, is basically rational number based. One interpretation would be that space-time surfaces, the induced spinors at space-time surfaces, configuration space spinor fields, S-matrix, etc..., can be obtained by algebraically continuing their values in a discrete subset of rational variant of the geometric structure considered to appropriate completion of rationals (real or p-adic). The existence of the algebraic continuation poses very strong additional constraints on physics but has not provided any practical means to solve quantum TGD.

In the following it is however demonstrated that this view leads to a very powerful iterative method of constructing global solutions of classical field equations from local data and at the same time gives justification for the notion of p-adic fractality, which has provided very successful approach not only to elementary particle physics but also physics at longer scales. The basic idea is that mere p-adic continuity and smoothness imply fractal long range correlations between rational points which are very close p-adically but far from each other in the real sense and vice versa.

3.1.1 The emergence of a rational cutoff

For a given p-adic continuation only a subset of rational points is acceptable since the simultaneous requirements of real and p-adic continuity can be satisfied only if one introduces ultraviolet cutoff length scale. This means that the distances between subset of rational points fixing the dynamics of the quantities involved are above some cutoff length scale, which is expected to depend on the p-adic number field R_p as well as a particular solution of field equations. The continued quantities coincide only in this subset of rationals but not in shorter length scales.

The presence of the rational cutoff implies that the dynamics at short scales becomes effectively discrete. Reality is however not discrete: discreteness and rationality only characterize the inherent limitations of our knowledge about reality. This conforms with the fact that our numerical calculations are always discrete and involve finite set of points.

The intersection points of various p-adic continuations with real space-time surface should code for all actual information that a particular p-adic physics can give about real physics in classical sense. There are reasons to believe that real space-time sheets are in the general case characterized by integers n decomposing into products of powers of primes p_i. One can expect that for p_i-adic continuations the sets of intersection points are especially large and that these p-adic space-time surfaces can be said to provide a good discrete cognitive mimicry of the real space-time surface.

Adelic formula represents real number as product of inverse of its p-adic norms. This raises the hope that taken together these intersections could allow to determine the real surface and thus classical physics to a high degree. This idea generalizes to quantum context too.

The actual construction of the algebraic continuation from a subset of rational points is of course something which cannot be done in practice and this is not even necessary since much more elegant approach is possible.

3.1.2 Hierarchy of algebraic physics

One of the basic hypothesis of quantum TGD is that it is possible to define exponent of Kähler action in terms of fermionic determinants associated with the modified Dirac operator derivable from a Dirac action related super-symmetrically to the Kähler action.

If this is true, a very elegant manner to define hierarchy of physics in various algebraic extensions of rational numbers and p-adic numbers becomes possible. The observation is that the continuation to various p-adic numbers fields and their extensions for the fermionic determinant can be simply done by allowing only the eigenvalues which belong to the extension of rationals involved and solve field equations for the resulting Kähler function. Hence a hierarchy of fermionic determinants results. The value of the dynamical Planck constant characterizes in this approach the scale factor of the M^4 metric in various number theoretical variants of the imbedding space $H = M^4 \times CP_2$ glued together along subsets of rational points of H. The values of $\hbar$ are determined from the requirement of quantum criticality [C6] meaning that Kähler coupling strength is analogous to critical temperature.

In this approach there is no need to restrict the imbedding space points to the algebraic extension of rationals and to try to formulate the counterparts of field equations in these discrete imbedding spaces.

3.1.3 p-Adic short range physics codes for long range real physics and vice versa

One should be able to construct global solutions of field equations numerically or by engineering them from the large repertoire of known exact solutions [D1]. This challenge looks formidable since the field equations are extremely non-linear and the failure of the strict non-determinism seems to make even in principle the construction of global solutions impossible as a boundary value problem or initial value problem.

The hope is that short distance physics might somehow code for long distance physics. If this kind of coding is possible at all, p-adicity should be crucial for achieving it. This suggests that one must articulate the question more precisely by characterizing what we mean with the phrases "short distance" and "long distance". The notion of short distance in p-adic physics is completely different from that in real physics, where rationals very close to each other can be arbitrary far away in the real sense, and vice versa. Could it be that in the statement "Short length scale physics codes for long length scale physics" the attribute "short"/"long" could refer to p-adic/real norm, real/p-adic norm, or both depending on the situation?

The point is that rational imbedding space points very near to each other in the real sense are in general at arbitrarily large distances in p-adic sense and vice versa. This observation leads to an elegant method of constructing solutions of field equations.

a) Select a rational point of the imbedding space and solve field equations in the real sense in an arbitrary small neighborhood U of this point. This can be done with an arbitrary accuracy by choosing U to be sufficiently small. It is possible to solve the linearized field equations or use a piece of an exact solution going through the point in question.

b) Select a subset of rational points in U and interpret them as points of p-adic imbedding space and space-time surface. In the p-adic sense these points are in general at arbitrary large distances from each and real continuity and smoothness alone imply p-adic long range correlations. Solve now p-adic field equations in p-adically small neighborhoods of these points. Again the accuracy can be arbitrarily high if the neighborhoods are choose small enough. The use of exact solutions of course allows to overcome the numerical restrictions.

c) Restrict the solutions in these small p-adic neighborhoods to rational points and interpret these points as real points having arbitrarily large distances. p-Adic smoothness and continuity alone imply fractal long range correlations between rational points which are arbitrary distant in the real sense. Return to a) and continue the loop indefinitely.

In this manner one obtains even in numerical approach more and more small neighborhoods representing almost exact p-adic and real solutions and the process can be continued indefinitely.

Some comments about the construction are in order.

a) Essentially two different field equations are in question: real field equations fix the local behavior of the real solutions and p-adic field equations fix the long range behavior of real solutions. Real/p-adic global behavior is transformed to local p-adic/real behavior. This might be the deepest reason why for the hierarchy of p-adic physics.

b) The failure of the strict determinism for the dynamics dictated by Kähler action and p-adic non-determinism due to the existence of p-adic pseudo constants give good hopes that the construction indeed makes it possible to glue together the (not necessarily) small pieces of space-time surfaces inside which solutions are very precise or exact.

c) Although the full solution might be impossible to achieve, the predicted long range correlations implied by the p-adic fractality at the real space-time surface are a testable prediction for which p-adic mass calculations and applications of TGD to biology provide support.

d) It is also possible to generalize the procedure by changing the value of p at some rational points and in this manner construct real space-time sheets characterized by different p-adic primes.

e) One can consider also the possibility that several p-adic solutions are constructed at given rational point and the rational points associated with p-adic space-time sheets labelled by $p_1, ..., p_n$ belong to the real surface.

This would mean that real surface would be multi-p p-adic fractal.

I have earlier suggested that even elementary particles are indeed characterized by integers and that only particles for which the integers have common prime factors interact by exchanging particles characterized by common prime factors. In particular, the primes $p = 2, 3, ..., 23$ would be common to the known elementary particles and appear in the expression of the gravitational constant. Multi-p p-fractality leads also to an explanation for the weakness of the gravitational constant. The construction recipe for the solutions would give a concrete meaning for these heuristic proposals.

This approach is not restricted to space-time dynamics but is expected to apply also at the level of say S-matrix and all mathematical object having physical relevance. For instance, p-adic four-momenta appear as parameters of S-matrix elements. p-Adic four-momenta very near to each other p-adically restricted to rational momenta define real momenta which are not close to each other and the mere p-adic continuity and smoothness imply fractal long range correlations in the real momentum space and vice versa.

3.1.4 p-Adic length scale hypothesis

Approximate p_1-adicity implies also approximate p_2-adicity of the space-time surface for primes $p \simeq p_1^k$. p-Adic length scale hypothesis indeed states that primes $p \simeq 2^k$ are favored and this might be due to simultaneous $p \simeq 2^k$- and 2-adicity. The long range fractal correlations in real space-time implied by 2-adicity would indeed resemble those implied by $p \simeq 2^k$ and both $p \simeq 2^k$-adic and 2-adic space-time sheets have larger number of common points with the real space-time sheet.

If the scaling factor λ of $\hbar$ appearing in the dark matter hierarchy is in good approximation $\lambda = 2^{11}$ also dark matter hierarchy comes into play in a resonant manner and dark space-time sheets at various levels of the hierarchy tend to have many intersection points with each other.

There is however a problem involved with the understanding of the origin of the p-adic length scale hypothesis if the correspondence via common rationals is assumed.

a) The mass calculations based on p-adic thermodynamics for Virasoro generator L_0 predict that mass squared is proportional to $1/p$ and Uncertainty Principle implies that L_p is proportional to $\sqrt{p}$ rather than p, which looks more natural if common rationals define the correspondence between real and p-adic physics.

b) It would seem that length $d_p \simeq pR$, R or order CP_2 length, in the induced space-time metric must correspond to a length $L_p \simeq \sqrt{p}R$ in M^4.

This could be understood if space-like geodesic lines at real space-time sheet obeying effective p-adic topology are like orbits of a particle performing Brownian motion so that the space-like geodesic connecting points with M^4 distance r_{M^4} has a length $r_{X^4} \propto r_{M^4}^2$. Geodesic random walk with randomness associated with the motion in CP_2 degrees of freedom could be in question. The effective p-adic topology indeed induces a strong local wiggling in CP_2 degrees of freedom so that r_{X^4} increases and can depend non-linearly on r_{M^4}.

c) If the size of the space-time sheet associated with the particle has size $d_p \sim pR$ in the induced metric, the corresponding M^4 size would be about $L_p \propto \sqrt{p}R$ and p-adic length scale hypothesis results.

d) The strongly non-perturbative and chaotic behavior $r_{X^4} \propto r_{M^4}^2$ is assumed to continue only up to L_p. At longer length scales the space-time distance d_p associated with L_p becomes the unit of space-time distance and geodesic distance r_{X^4} is in a good approximation given by

$$r_{X^4} = \frac{r_{M^4}}{L_p} d_p \propto \sqrt{p} \times r_{M^4} \ , \tag{1}$$

and is thus linear in M^4 distance r_{M^4}.

3.1.5 How to understand the value of gravitational constant?

The proposed explanation of the p-adic length scale hypothesis allows also to understand the weakness of the gravitational constant as being due to the fact that the space-time distance r_{X^4} appearing in gravitational force as given by Newtonian approximation is by a factor $\sqrt{p}$ times longer than r_{M^4} so that the strong gravitational force proportional to $L_p^2/r_{X^4}^2$ scales down by a factor p as r_{X^4} is expressed in terms of r_{M^4}. M^4 distance r_{M^4} is indeed the natural variable since distances are measured using space-time sheets as units and their sizes are always measured in M^4 metric or almost flat X^4 metric.

A more precise argument goes as follows.

a) Assume first that the space-time sheet is characterized by single prime p: also multi-p-fractality is possible. The strong gravitational constant G_s characterizes the interactions involving exchanges of string like objects of size scale measured naturally using L_p as a unit. In this case the force is mediated in M^4 as an exchange of a particle. By dimensional estimate G_s is proportional to L_p^2 and string model picture gives a precise estimate for the numerical factor n in $G_s = nL_p^2$.

b) The classical gravitational force is mediated via induced metric inside the space-time sheets and in long length scales is proportional to $G_s/r^2_{X^4} \propto L^2_p/r^2_{X^4} \propto R^2/r^2_{M^4}$, where $R \simeq 10^4\sqrt{G}$ is CP_2 length. Hence the effective gravitational constant is reduced by a factor $1/p$ and is same for all values of p.

c) The value of the gravitational constant is still by a factor of order $R^2/G \sim 10^8$ too high. A correct value is obtained if multi-p-fractality prevails in such a manner that $\sqrt{p}$ is replaced by $\sqrt{n}$ with $n = 2 \times 3 \times 5... \times 23 \times p$. On can visualize the situation as hierarchy of wavelets: to p-adic wavelets very small $q = 23$-adic wawelets are superposed to which in turn $q = 19$-adic ... The earlier estimates for the gravitational constant are consistent with this result and fix the numerical details.

d) This approach predicts a p-adic hierarchy of strong gravitons and unstable spin 2 hadrons are excellent candidates for them. It is however not clear whether Newtonian graviton is predicted at all: in other words could the gravitation inside space-time sheets be a purely classical phenomenon? One can certainly imagine the exchange of topologically condensed Newtonian gravitons moving along light-like geodesics along space-time sheets and the lengths of spatial projections of these geodesics would be indeed scaled up by $\sqrt{p}$ factor.

3.1.6 But what about gauge coupling constants?

The previous argument is very nice but applied to gauge coupling strengths would give the same result! A possible way out is that topologically condensed gauge bosons move along geodesic lines only below the p-adic length scale $L(k_1)$, $k_1 \ll p$. This would mean that the length scale below which $r(X^4) \propto r^2_{M^4}$ holds true depends on the topologically condensed boson. This scale should be logarithmically shorter for gauge bosons than for gravitons at a space-time sheet characterized by prime p.

This would suggest that the unit for length is such that the distance along X^4 satisfies the fractal relationship

$$\begin{aligned} r^2_{X^4} &= k_1 r^2_{M^4} \ , \\ k_1 &= k + k_0 \equiv log_2(p) \ , \end{aligned} \tag{2}$$

where k_0 is an integer (which could be also negative). k_1 could characterize the size of the gauge boson. This implies that the gauge force corresponds to a renormalized coupling constant strength $\alpha_1/(log_2(p) + k_0)$. This kind of formula follows for Kähler coupling strength from the p-adic constancy of G.

3.1.7 Does cognition automatically solve real field equations in long length scales?

In TGD inspired theory of consciousness p-adic space-time sheets are identified as space-time correlates of cognition. Therefore our thoughts would have literally infinite size in the real topology if p-adics and reals correspond to each other via common rationals (also other correspondence based on the separate canonical identification of integers m and n in $q = m/n$ with p-adic numbers).

The cognitive solution of field equations in very small p-adic region would solve field equations in real sense in a discrete point set in very long real length scales. This would allow to understand why the notions of Universe and infinity are a natural part of our conscious experience although our sensory input is about an infinitesimally small region in the scale of universe.

The idea about Universe performing mimicry at all possible levels is one of the basic ideas of TGD inspired theory of consciousness. Universe could indeed understand and represent the long length scale real dynamics using local p-adic physics. The challenge would be to make quantum jumps generating p-adic surfaces having large number of common points with the real space-time surface. We are used to call this activity theorizing and the progress of science towards smaller real length scales means progress towards longer length scales in p-adic sense. Also real physics can represent p-adic physics: written language and computer represent examples of this mimicry.

3.2 A more detailed view about how local p-adic physics codes for p-adic fractal long range correlations of the real physics

The vision just described gives only a rough heuristic view about how the local p-adic physics could code for the p-adic fractality of long range real physics. There are highly non-trivial details related to the treatment of M^4 and CP_2 coordinates and to the mapping of p-adic H-coordinates to their real counterparts and vice versa.

3.2.1 How real and p-adic space-time regions are glued together?

The first task is to visualize how real and p-adic space-time regions relate to each other. It is convenient to start with the extension of real axis to contain also p-adic points. For finite rationals $q = m/n$, m and n have finite power expansions in powers of p and one can always write $q = p^k \times r/s$ such that r and s are not divisible by p and thus have pinary expansion of in powers of p as $x = x_0 + \sum_1^N x_n p^n$, $x_i \in \{0, p\}$, $x_0 \neq 0$.

One can always express p-adic number as $x = p^n y$ where y has p-adic norm 1 and has expansion in non-negative powers of p. When x is rational but not integer the expansion contains infinite number of terms but is periodic. If the expansion is infinite and non-periodic, one can speak about *strictly p-adic* number having infinite value as a real number.

In the same manner real number x can be written as $x = p^n y$, where y is either rational or has infinite non-periodic expansion $y = r_0 + \sum_{n>0} r_n p^{-n}$ in negative powers of p. As a p-adic number y is infinite. In this case one can speak about strictly real numbers.

This gives a visual idea about what the solution of field equations locally in various number fields could mean and how these solutions are glued together along common rationals. In the following I shall be somewhat sloppy and treat the rational points of the imbedding space as if they were points of real axis in order to avoid clumsy formulas.

a) The p-adic variants of field equations can be solved in the strictly p-adic realm and by p-adic smoothness these solutions are well defined also in as subset of rational points. The strictly p-adic points in a neighborhood of a given rational point correspond as real points to infinitely distant points of M^4. The possibility of p-adic pseudo constants means that for rational points of M^4 having sufficiently large p-adic norm, the values of CP_2 coordinates or induced spinor fields can be chosen more or less freely.

b) One can solve the p-adic field equations in any p-adic neighborhood $U_n(q) = \{x = q + p^n y\}$ of a rational point q of M^4, where y has a unit p-adic norm and select the values of fields at different points q_1 and q_2 freely as long as the spheres $U_n(q_1)$ and $U_n(q_2)$ are disjoint (these spheres are either identical or disjoint by p-adic ultra-metricity).

The points in the p-adic continuum part of these solutions are at an infinite distance from q in M^4. The points which are well-defined in real sense form a discrete subset of rational points of M^4. The p-adic space-time surface constructed in this manner defines a discrete fractal hierarchy of rational space-time points besides the original points inside the p-adic spheres. In real sense the rational points have finite distances and could belong to disjoint real space-time sheets. The failure of the strict non-determinism for the field equations in the real sense gives hopes for gluing these sheets partially together (say in particle reactions with particles represented as 3-surfaces).

c) All rational points q of the p-adic space-time sheet can be interpreted as real rational points and one can solve the field equations in the real sense in the neighborhoods $U_n(q) = \{x = q + p^n y\}$ corresponding to real numbers in the the range $p^n \leq x \leq p^{n+1}$. Real smoothness and continuity fix the solutions at finite rational points inside $U_n(q)$ and by the phenomenon of p-adic pseudo constants these values can be consistent with p-adic field equations. Obviously one can can continue the construction process indefinitely.

3.2.2 p-Adic scalings act only in M^4 degrees of freedom

p-Adic fractality suggests that finite real space-time sheets around points $x + p^n$, $x = 0$, are obtained as by just scaling of the M^4 coordinates having origin at $x = 0$ by p^n of the solution defined in a neighborhood of x and leaving CP_2 coordinates as such. The known extremals of Kähler action indeed allow M^4 scalings as dynamical symmetries.

One can understand why no scaling should appear in CP_2 degrees of freedom. CP_2 is complex projective space for which points can be regarded as complex planes and for these p-adic scalings act trivially. It is worth of emphasizing that here could lie a further deep number theoretic reason for why the space S in $H = M^4 \times S$ must be a projective space.

3.2.3 What p-adic fractality for real space-time surfaces really means?

The identification of p-adic and real M^4 coordinates of rational points as such is crucial for p-adic fractality. On the other hand, the identification rational real and p-adic CP_2 coordinates as such would not be consistent with the idea that p-adic smoothness and continuity imply p-adic fractality manifested as long range correlations for real space-time sheets

The point is that p-adic fractality is not stable against small p-adic deformations of CP_2 coordinates as function of M^4 coordinates for solutions representable as maps $M^4 \rightarrow CP_2$. Indeed, if the rational valued p-adic CP_2 coordinates are mapped as such to real coordinates, the addition of large power p^n to CP_2 coordinate implies small modification in p-adic sense but large change in the real sense so that correlations of CP_2 at p-adically scaled M^4 points would be completely lost.

The situation changes if the map of p-adic CP_2 coordinates to real ones is continuous so that p-adically small deformations of the p-adic space-time points are mapped to small real deformations of the real space-time points.

a) Canonical identification $I : x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ satisfies continuity constraint but does not map rationals to rationals.

b) The modification of the canonical identification given by

$$I(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{3}$$

is uniquely defined for rational points, maps rationals to rationals, has a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$.

c) The form of this map is not general coordinate invariant nor invariant under color isometries. The natural requirement is that the map should respect the symmetries of CP_2 maximally. Therefore the complex coordinates transforming linearly under U(2) subgroup of SU(3) defining the projective coordinates of CP_2 are a natural choice. The map in question would map the real components of complex coordinates to their p-adic variants and vice versa. The residual $U(2)$ symmetries correspond to rational unitary 2×2-matrices for which matrix elements are of form $U_{ij} = p^k r/s$, $r < p, s < p$. It would seem that these transformations must form a finite subgroup if they define a subgroup at all. In case of $U(1)$ Pythagorean phases define rational phases but sufficiently high powers fail to satisfy the conditions $r < p, s < p$. Also algebraic extensions of p-adic numbers can be considered.

d) The possibility of pseudo constant allows to modify canonical identification further so that it reduces to the direct identification of real and p-adic rationals if the highest powers of p in r and s ($q = p^n r/s$) are not higher than p^N. Write $x = \sum_{n \geq 0} x_n p^n = x^{N)} + p^{N+1} y$ with $x^{N)} = \sum_{n=0}^{N} x_n p^n$, $x_0 \neq 0$, $y_0 \neq 0$, and define $I_N(x) = x^{N)} + p^{N+1} I(y)$. For $q = p^n r/s$ define $I_N(q) = p^n I_N(r)/I_N(s)$. This map reduces to the direct identification of real and p-adic rationals for $y = 0$.

e) There is no need to introduce the imaginary unit explicitly. In case of spinors imaginary unit can be represented by the antisymmetric 2×2-matrix ϵ_{ij} satisfying $\epsilon_{12} = 1$. As a matter fact, the introduction of imaginary unit as number would lead to problems since for $p \ mod \ 4 = 3$ imaginary unit should be introduced as an algebraic extension and CP_2 in this sense would be an algebraic extension of RP_2. The fact that the algebraic extension of p-adic numbers by $\sqrt{-1}$ is equivalent with an extension introducing $\sqrt{p-1}$ supports the view that algebraic imaginary unit has nothing to do with the geometric imaginary unit defined by Kähler form of CP_2. For $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as a p-adic number but is infinite as a real number so that the notion of finite complex rational would not make sense.

3.2.4 Preferred CP_2 coordinates as a space-time correlate for the selection of quantization axis

Complex CP_2 coordinates are fixed only apart from the choice of the quantization directions of color isospin and hyper charge axis in SU(3) Lie algebra. Hence the selection of quantization axes seems to emerge at the level of the generalized space-time geometry as quantum classical correspondence indeed requires.

In a well-defined sense the choice of the quantization axis and a special coordinate system implies the breaking of color symmetry and general coordinate invariance. This breaking is induced by the presence of p-adic space-time sheets identified as correlates for cognition and intentionality. One could perhaps say that the cognition affects real physics via the imbedding space points shared by real and p-adic space-time sheets and that these common points define discrete coordinatization of the real space-time surface analogous to discretization resulting in any numerical computation.

3.2.5 Relationship between real and p-adic induced spinor fields

Besides imbedding space coordinates also induced spinor fields are fundamental variables in TGD. The free second quantized induced spinor fields define the fermionic oscillator operators in terms of which the gamma matrices giving rise to spinor structure of the "world of classical worlds" can be expressed.

p-Adic fractal long range correlations must hold true also for the induced spinor fields and they are in exactly the same role as CP_2 coordinates so that the variant of canonical identification mapping rationals to rationals should map the real and imaginary parts of of real induced spinor fields to their p-adic counterparts and vice versa at the rational space-time points common to p-adic and real space-time sheets.

3.2.6 Could quantum jumps transforming intentions to actions really occur?

The idea that intentional action corresponds to a quantum jump in which p-adic space-time sheet is transformed to a real one traversing through rational points common to p-adic and real space-time sheet is consistent with the conservation laws since the sign of the conserved inertial energy can be also negative in TGD framework and the density of inertial energy vanishes in cosmological length scales [D5]. Also the non-diagonal transitions $p_1 \rightarrow p_2$ are in principle possible and would correspond to intersections of p-adic space-time sheets having a common subset of rational points. Kind of phase transitions changing the character of intention or cognition would be in question.

1. Realization of intention as a scattering process

The first question concerns the interpretation of this process and possibility to find some familiar counterpart for it in quantum field theory framework. The general framework of quantum TGD suggests that the points common to real and p-adic space-time sheets could perhaps be regarded as arguments of an n-point function determining the transition amplitudes for p-adic to real transition or $p_1 \rightarrow p_2$-adic transitions. The scattering event transforming an p-adic surface (infinitely distant real surface in real M^4) to a real finite sized surface (infinitely distant p-adic surface in p-adic M^4) would be in question.

2. Could S-matrix for realizations of intentions have the same general form as the ordinary S-matrix?

One might hope that the realization of intention as a number theoretic scattering process could be characterized by an S-matrix, which one might hope of being unitary in some sense. These S-matrix elements could be interpreted at fundamental level as probability amplitudes between intentions to prepare a define initial state and the state resulting in the process.

Super-conformal invariance is a basic symmetry of quantum TGD which suggests that the S-matrix in question should be constructible in terms of n-point functions of a conformal field theory restricted to a subset of rational points shared by real and p-adic space-time surfaces or their causal determinants. According to the general vision discussed in [C1], the construction of n-point functions effectively reduces to that at 2-dimensional sections of light-like causal determinants of space-time surfaces identified as partonic space-time sheets.

The idea that physics in various number fields results by algebraic continuation of rational physics serves as a valuable guideline and suggests that the form of the S-matrices between different number fields (call them non-diagonal S-matrices) could be essentially the same as that of diagonal S-matrices. If this picture is correct then the basic differences to ordinary real S-matrix would be following.

a) Intentional action could transform p-adic space-time surface to a real one only if the exponent of Kähler function for both is rational valued (or belongs to algebraic extension of rationals).

b) The points appearing as arguments of n-point function associated with the non-diagonal S-matrix are a subset of rational points of imbedding space whereas in the real case, where the integration over these points is well defined, all values of arguments can be allowed. Thus the difference between ordinary S-matrix and more general S-matrices would be that a continuous Fourier transform of n-point function in space-time domain is not possible in the latter case. The inherent nature of cognition would be that it favors localization in the position space.

3. Objection and its resolution

Exponent of Kähler function is the key piece of the configuration space spinor field. There is a strong counter argument against the existence of the Kähler function in the p-adic context. The basic problem is that the definite integral defining the Kähler action is not p-adically well-defined except in the special cases when it can be done algebraically. Algebraic integration is however very tricky and numerically completely unstable.

The definition of the exponent of Kähler function in terms of Dirac determinants or, perhaps equivalently, as a result of normal ordering of the modified Dirac action for second quantized induced spinors might however lead to an elegant resolution of this problem. This approach is discussed in detail in [B4, D1]. The idea is that Dirac determinant can be defined as a product of eigenvalues of the modified Dirac operator and one ends up to a hierarchy of theories based on the restriction of the eigenvalues to various algebraic extensions of rationals identified as a hierarchy associated with corresponding algebraic extensions of p-adic numbers. This hierarchy corresponds to a hierarchy of theories (and also physics!) based on varying values of Kähler coupling constant and Planck constant. The elegance of this approach is that no discretization at space-time level would be needed: everything reduces to the generalized eigenvalue spectrum of the modified Dirac operator.

4. A more detailed view

Consider the proposed approach in more detail.

a) Fermionic oscillator operators are assigned with the generalized eigenvectors of the modified Dirac operator defined at the light-like causal determinants:

$$\begin{aligned} \Psi &= \sum_n \Psi_n b_n \ , \\ D\Psi_n &= \Gamma^\alpha D_\alpha \Psi_n = \lambda_n O \Psi_n \ , \ \ O \equiv n_\alpha \Gamma^\alpha \ . \end{aligned} \quad (4)$$

Here $\Gamma^\alpha = T^{\alpha k}\Gamma_k$ denote so called modified gamma matrices expressible in terms of the energy momentum current $T^{\alpha k}$ assignable to Kähler action [B4]. The replacement of the ordinary gamma matrices with modified ones is forced by the requirement that the super-symmetries of the modified Dirac action are consistent with the property of being an extremal of Kähler action. n_α is a light like vector assignable to the light-like causal determinant and $O = n_\alpha\Gamma^\alpha$ must be rational and have the same value at real and p-adic side at rational points. The integer n labels the eigenvalues λ_n of the modified Dirac operator, and b_n corresponds to the corresponding fermionic oscillator operator.

b) The condition that the p-adic and real variants Ψ if the Ψ are identical at common rational points of real and p-adic space-time surface (the same applies to 4-surfaces corresponding to different p-adic number fields) poses a strong constraint on the algebraic continuation from rationals to p-adics and gives hopes of deriving implications of this approach.

c) Ordinary fermionic anti-commutation relations do not refer specifically to any number field. Super Virasoro (anti-)commutation relations involve only rationals. This suggest that fermionic Fock space spanned by the oscillator operators b_n is universal and same for reals and p-adic numbers and can be regarded as rational. Same would apply to Super Virasoro representations. Also the possibility to interpret configuration space spinor fields as quantum superpositions of Boolean statements supports this kind of universality. This gives good hopes that the contribution of the inner produces between Fock states to the S-matrix elements are number field independent.

d) Dirac determinant can be defined as the product of the eigenvalues λ_n restricted to a given algebraic extension of rationals. The solutions of the modified Dirac equation correspond to vanishing eigen values and define zero modes generating conformal super-symmetries and are not of course included.

e) Only those operators b_n for which λ_n belongs to the algebraic extension of rationals in question are used to construct physical states for a given algebraic extension of rationals. This might mean an enormous simplification of the formalism in accordance with the fact that configuration space Clifford algebra corresponds as a von Neumann algebra to a hyper-finite factor of type II_1 for which finite truncations by definition allow excellent approximations [C6]. One can even ask whether this hierarchy of algebraic extensions of rationals could in fact define a hierarchy of finite-dimensional Clifford algebras. If so then the general theory of hyper-finite factors of type II_1 would provide an extremely powerful tool.

3.3 Cognition, logic, and p-adicity

There seems to be a nice connection between logic aspects of cognition and p-adicity. In particular, p-valued logic for $p = 2^k - n$ has interpretation in terms of ordinary Boolean logic with n "taboos" so that p-valued logic does not conflict with common sense in this case. Also an interpretation of projections of p-adic space-time sheets to an integer lattice of real Minkowski space M^4 in terms of generalized Boolean functions emerges naturally so that M^4 projections of p-adic space-time would represent Boolean functions for a logic with n taboos.

3.3.1 2-adic valued functions of 2-adic variable and Boolean functions

The binary coefficients f_{nk} in the 2-adic expansions of terms $f_n x^n$ in the 2-adic Taylor expansion $f(x) = \sum_{n=0} f_n x^n$, assign a sequence of truth values to a 2-adic integer valued argument $x \in \{0, 1, ..., 2^N\}$ defining a sequence of N bits. Hence $f(x)$ assigns to each bit of this sequence a sequence of truth values which are ordered in the sense that the truth values corresponding to bits are not so important p-adically: much like higher decimals in decimal expansion. If a binary cutoff in N:th bit of $f(x)$ is introduced, B^M-valued function in B^N results, where B denotes Boolean algebra fo 2 elements. The formal generalization to p-adic case is trivial: 2 possible truth values are only replaced by p truth values representable as $0, ..., p-1$.

3.3.2 p-Adic valued functions of p-adic variable as generalized Boolean functions

One can speak of a generalized Boolean function mapping finite sequences of p-valued Boolean arguments to finite sequences of p-valued Boolean arguments. The restriction to a subset $x = kp^n$, $k = 0, ..., p-1$ and the replacement of the function $f(x)$ with its lowest pinary digit gives a generalized Boolean function of a single p-valued argument. If $f(x)$ is invariant under the scalings by powers of p^k, one obtains a hologram like representation of the generalized Boolean function with same function represented in infinitely many length scales. This guarantees the robustness of the representation.

The special role of 2-adicity explaining p-adic length scale hypothesis $p \simeq 2^k$, k integer, in terms of multi-p-acidic fractality would correlate with the special role of 2-valued logic in the world order. The fact that all generalizations of 2-valued logic ultimately involve 2-adic logic at the highest level, where the generalization is formulated would be analog of p-adic length scale hypothesis.

3.3.3 $p = 2^k - n$-adicity and Boolean functions with taboos

It is difficult to assign any reasonable interpretation to $p > 2$-valued logic. Also the generalization of logical connectives AND and OR is far from obvious. In the case $p = 2^k - n$ favored by the p-adic length scale hypothesis situation is however different. In this case one has interpretation in terms B^k with n Boolean statements dropped out so that one obtains what might be called $\hat{B^k}$. Since n is odd this set is not invariant under Boolean conjugation so that there is at least one statement, which is identically true and could be called taboo, axiom, or dogma: depending on taste. The allowed Boolean functions would be constructed in this case using standard Boolean functions AND and OR with the constraint that taboos are respected: in other words, both the inputs and values of functions belong to $\hat{B^k}$.

A unique manner to define the logic with taboos is to require that the number of taboos is maximal so that if statement is dropped its negation remains in the logic. This implies $n > B^k/2$.

3.3.4 The projections of p-adic space-time sheets to real imbedding space as representations of of Boolean functions

Quantum classical correspondence suggests that generalized Boolean functions should have space-time correlates. Since Boolean cognition involves free will, it should be possible to construct space-time representations of arbitrary Boolean functions with finite number of arguments freely. The non-determinism of p-adic differential equations guarantees this freedom.

p-Adic space-time sheets and p-adic non-determinism make possible to represent generalization of Boolean functions of four Boolean variables obtained by replacing both argument and function with p-valued pinary digit instead of bit. These representations result as discrete projections of p-adic space-time sheets to integer valued points of real Minkowski space M^4. The interpretation would be in terms of 4 sequences of truth values of p-valued logic associated with a finite 4-D integer lattice whose lattice points can be identified as sequences of truth values of a p-valued logic with a set of p-valued truth value at each point so that in the 2-adic case one has map $B^{4M} \to B^{4N}$. Here the number of lattice points in a given coordinate direction of M^4 is M and N is the number of bits allowed by binary cutoff for CP_2 coordinates. For $p = 2^k - n$ representing Boolean algebra with n taboos, the maps can be interpreted as maps $\hat{B}^{4M} \to \hat{B}^{4N}$.

These lattices can be seen as subsets of rational shadows of p-adic space-time sheets to Minkowski space. The condensed matter analog would be a lattice with a a sequence of p-valued dynamical variables (sequence of bits/spins for $p = 2$) at each lattice point. At a fixed spatial point of M^4 the lowest bits define a time evolution of a generalized Boolean function: $B \to B$.

These observations support the view that intentionality and logic related cognition could perhaps be regarded as 2-adic aspects of consciousness. The special role of primes $p = 2^k - n$ could also be understood as special role of Boolean logic among p-valued logics and $p = 2^k - n$ logic would correspond to B^k with n axioms representing logic respecting a belief system with n beliefs. Recall that multi-p p-adic fractality involving 2-adic fractality is possible for the solutions of field equations and explains p-adic length scale hypothesis.

Most points of the p-adic space-time sheets correspond to real points which are literally infinite as real points. Therefore cognition would be in quite literal sense outside the real cosmos. Perhaps this is a direct correlate for the basic experience that mind is looking the material world from outside.

3.3.5 Connection with the theory of computational complexity?

There are interesting questions concerning the interpretation of four generalized Boolean arguments. TGD explains the number $D = 4$ for space-time dimensions and also the dimension of imbedding space. Could one also find explanation why $d = 4$ defines special value for the number of generalized Boolean inputs and outputs?

a) Could the general theory of computational complexity allow to understand $d = 4$ as a maximum number of inputs and outputs allowing the computation of something related to these functions in polynomial time? For instance, complexity theorist could probably immediately answer following questions. Could the computation of the 2-adic values of CP_2 coordinates as a function of 2-adic M^4 coordinates expressed in terms of fundamental logical connectives take a time which is polynomial as a function of the number of N^4 pinary digits of M^4 coordinates and N^4 pinary digits of CP_2 coordinates? Is this time non-polynomial for M^d and S_d, S_d d-dimensional internal space, $d > 4$. Unfortunately I do not possess the needed complexity theoretic knowhow to answer these questions.

b) The same question could make sense also for $p > 2$ if the notion of the logical connectives and functions generalizes as it indeed does for $p = 2^k - n$.

Therefore the question would be whether p-adic length scale hypothesis and dimensions of imbedding space and space-time are implied by a polynomial computation time? This could be the case since essentially a restriction of values and arguments of Boolean functions to a subset of B^k is in question.

3.3.6 Some calculational details

In the following the details of p-adic non-determinism are described for a differential equation of single p-adic variable and some comments about the generalization to the realistic case are given.

1. One-dimensional case

To understand the essentials consider for simplicity a solution of a p-adic differential equation giving function $y = f(x)$ of one independent variable $x = \sum_{n \geq n_0} x_n p^n$.

a) p-Adic non-determinism means that the initial values $f(x)$ of the solution can be fixed arbitrarily up to $N+1$:th pinary digit. In other words, $f(x_N)$, where $x_N = \sum_{n_0 \leq n \geq N} x_n p^n$ is a rational obtained by dropping all pinary digits higher than N in $x = \sum_{n \geq n_0} x_n p^n$ can be chosen arbitrarily.

b) Consider the projection of $f(x)$ to the set of rationals assumed to be common to reals and p-adics.

i) Genuinely p-adic numbers have infinite number of positive pinary digits in their non-periodic expansion (non-periodicity guarantees non-rationality) and are strictly infinite as real numbers. In this regime p-adic differential equation fixes completely the solution. This is the case also at rational points $q = m/n$ having infinite number of pinary digits in their pinary expansion.

ii) The projection of p-adic x-axis to real axis consists of rationals. The set in which solution of p-adic differential equations is non-vanishing can be chosen rather freely. For instance, p-adic ball of radius p^{-n} consisting of points $x = p^M y$, $y \neq 0$, $|y|_p \leq 1$, can be considered. Assume $N > M$. p-Adic nondeterminism implies that $f(q)$ for $q = \sum_{M \leq n \leq N} x_n p^n$, can be chosen arbitrarily. For $M \geq 0$ q is always integer valued and the scaling of x by a suitable power of p always allows to get a finite integer lattice at x-axis.

iii) The lowest pinary digit in the expansion of $f(q)$ in powers of p in defines a pinary digit. These pinary digits would define a representation for a sequence of truth values of p-logic. $p = 2$ gives the ordinary Boolean logic. It is also interpret this pinary function as a function of pinary argument giving Boolean function of one variable in 2-adic case.

2. Generalization to the space-time level

This picture generalizes to space-time level in a rather straight forward manner. y is replaced with CP_2 coordinates, x is replaced with M^4 coordinates, and differential equation with field equations deducible from the Kähler action. The essential point is that p-adic space-time sheets have projection to real Minkowski space which consists of a discrete subset of integers when suitable scaling of M^4 coordinates is allowed. The restriction of 4 CP_2 coordinates to a finite integer lattice of M^4 defines 4 Boolean functions of four Boolean arguments or their generalizations for $p > 2$. Also the modes of the induce spinor field define a similar representation.

4 TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts

This chapter is second part of the multi-chapter devoted to the vision about TGD as a generalized number theory. The basic theme of the chapter is that TGD allows two dual pictures about space-time as a 4-surface. In the first picture space-times are regarded as hyper-quaternionic 4-surfaces in 8-dimensional hyper-octonionic space $HO = M^8$. In the second picture space-times are regarded as 4-surfaces in $M^4 \times CP_2$ satisfying field equations guaranteing absolute minimization of Kähler action.

4.1 Development of ideas

The discussions for years ago with Tony Smith [2] stimulated very general ideas about space-time surface as an associative, quaternionic sub-manifold of octonionic 8-space. Also the observation that quaternionic and octonionic primes have norm squared equal to prime in complete accordance with p-adic length scale hypothesis, led to suspect that the notion of primeness for quaternions, and perhaps even for octonions, might be fundamental for the formulation of quantum TGD. The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional quaternions and octonions.

It took some years to realize that the difficulties related to the realization of Lorentz invariance might be overcome by replacing quaternions and octonions with hyper-quaternions and hyper-octonions. Hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$ and can be regarded as a sub-space of complexified quaternions *resp.* octonions. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity. The loss of number field and even sub-algebra property is not fatal and has a clear physical meaning. The notion of primeness is inherited from that for complexified quaternions *resp.* octonions.

The notion of hyper-quaternionic and octonionic manifold makes sense but it not plausible that $H = M^4 \times CP_2$ could be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces are assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with a fixed complex structure (containing preferred imaginary unit) are labelled by CP_2, each hyper-quaternionic and co-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs. What deserves separate emphasis is that the basic structure of the standard model would reduced to number theory.

Some notational conventions are in order before continuing. The fields of quaternions *resp.* octonions having dimension 4 *resp.* 8 and will be denoted by Q and O. Their complexified variants will be denoted by Q_C and O_C. The sub-spaces of hyper-quaternions HQ and hyper-octonions HO are obtained by multiplying the quaternionic and octonionic imaginary units by $\sqrt{-1}$. These sub-spaces are very intimately related with the corresponding algebras, and can be seen as Euclidian and Minkowkian variants of the same basic structure. Also the Abelianized versions of the hyper-quaternionic and -octonionic sub-spaces can be considered: these algebras have a representation in the space of spinors of imbedding space $H = M^4 \times CP_2$.

4.2 Space-time-surface as a hyper-quaternionic or co-hyper-quaternionic sub-manifold of hyper-octonionic imbedding space?

Space-time identified as a hyper-quaternionic sub-manifold of the hyper-octonionic space in the sense that the tangent space of the space-time surface defines a hyper-quaternionic sub-space of the hyper-octonionic tangent space of H at each space-time point, looks an attractive idea. Also co-hyper-quaternionic surfaces correspond to space-time surfaces. Second possibility is that the algebra generated by tangent space *resp.* normal space of the space-time surface is associative (associativity *resp.* co-associativity). Also the possibility that the dynamics of the space-time surface is determined from the requirement that space-time surface X^4 is algebraically closed in the sense that tangential or normal algebra at each point corresponds to a 4-D sub-algebra of complexified octonions.

Some delicacies are caused by the question whether the induced algebra at X^4 is just the hyper-octonionic product or whether the algebra product is projected to the space-time surface. If the normal part of the product is projected out, the space-time algebra closes automatically.

The first guess would be that space-time surfaces are hyper-quaternionic of co-hyper-quaternionic sub-manifolds of hyper-octonionic space $HO = M^8$ with the property that complex structure is fixed and same at all points of space-time surface. This corresponds to a global selection of a preferred octonionic imaginary unit. The automorphisms leaving this selection invariant form group $SU(3)$ identifiable as color group. The selections of (co-)hyper-quaternionic sub-space under this condition are parameterized by CP_2. This means that each 4-surface in HO defines a 4-surface in $M^4 \times CP_2$ and one can speak about number-theoretic analog of spontaneous compactification having of course nothing to do with dynamics. It would be possible to make physics in two radically different geometric pictures: HO picture and $H = M^4 \times CP_2$ picture.

For a theoretical physicists of my generation it is easy to guess that the next step is to realize that it is possible to fix the preferred octonionic imaginary at each point of HO separately so that local $S^6 = G_2/SU(3)$, or equivalently the local group G_2 subject to $SU(3)$ gauge invariance, characterizes the possible choices of (co-)hyper-quaternionic structure with a preferred imaginary unit. $G_2 \subset SO(7)$ is the automorphism group of octonions, and appears also in M-theory. This local choice has interpretation as a fixing of the plane of non-physical polarizations and rise to degeneracy which is a good candidate for the ground state degeneracy caused by the vacuum extremals.

$OH - -M^4 \times CP_2$ duality allows to construct a foliation of HO by (co-)hyper-quaternionic space-time surfaces in terms of maps $HO \rightarrow SU(3)$ satisfying certain integrability conditions guaranteing that the distribution of (co-)hyper-quaternionic planes integrates to a foliation by 4-surfaces. In fact, the freedom to fix the preferred imaginary unit locally extends the maps to $HO \rightarrow G_2$ reducing to maps $HO \rightarrow SU(3) \times S^6$ in the local trivialization of G_2. This foliation defines a four-parameter family of 4-surfaces in $M^4 \times CP_2$ for each local choice of the preferred imaginary unit. The dual of this foliation defines a 4-parameter family co-hyper-quaternionic space-time surfaces. HQ and $coHQ$ surfaces intersect generically in a finite number of points.

Hyper-octonion analytic functions $HO \rightarrow HO$ with real Taylor coefficients provide a physically motivated ansatz satisfying the integrability conditions. The basic reason is that hyper-octonion analyticity is not plagued by the complications due to non-commutativity and non-associativity. Indeed, this notion results also if the product is Abelianized by assuming that different octonionic imaginary units multiply to zero. A good candidate for the HO dynamics is free massless Dirac action with Weyl condition for an octonion valued spinor field using octonionic representation of gamma matrices and coupled to the G_2 gauge potential defined by the tensor 7×7 tensor product of the imaginary parts of spinor fields.

The basic conjecture is that the absolute minima of Kähler action in $H = M^4 \times CP_2$ correspond to the hyper-quaternion analytic surfaces in HO. The map $f : HO \rightarrow S^6$ would probably satisfy some constraints posed by the requirement that the resulting surfaces define solutions of field equations in $M^4 \times CP_2$ picture. This conjecture has several variants. It could be that only the asymptotic behavior corresponds to hyper-quaternion analytic function but that hyper-quaternionicity is a general property of preferred extrema of Kähler action. The encouraging hint is the fact that Hamilton-Jacobi coordinates coding for the local selection of the plane of non-physical polarizations, appear naturally also in the construction of general solutions of field equations [D1].

It will be found that hyper-quaternion analytic surfaces cannot correspond to the absolute minima of Kähler action. Rather the absolute value of the contribution from a region with given sign of action density is either minimized or maximized. The most obvious guess is that HQ *resp.* $coHQ$ surfaces correspond to minima *resp.* maxima for the absolute values of these contributions. In particular, small deformations of empty Minkowski space *resp.* CP_2 type extremals would correspond to HQ *resp.* $coHQ$ surfaces. Also the dual Kähler action defined by the projection of CP_2 Kähler form to the normal space of space-time surface could be the action principle defining $coHQ$ 4-surfaces as its preferred extrema.

4.3 The notion of Kähler calibration

Calibration is a closed p-form, whose value for a given p-plane is not larger than its volume in the induced metric. What is important that if it is maximum for tangent planes of p-sub-manifold, minimal surface with smallest volume in its homology equivalence class results.

The idea of Kähler calibration is based on a simple observation. Hyper-octonionic spinor field defines a map $M^8 \rightarrow H = M^4 \times CP_2$ allowing to induce metric and Kähler form of H to M^8. Also Kähler action is well defined for the local hyper-quaternion plane.

The idea is that the non-closed 4-form associated the wedge product of unit tangent vectors of hyper-quaternionic plane in M^8 and saturating to volume for it becomes closed by multiplication with Kähler action density L_K. If L_K is minimal for hyper-quaternion plane, HQ manifolds define extremals of Kähler action for which the magnitudes of positive and negative contributions to the action are separately minimized. If L_K is maximal it could correspond $coHQ$ surfaces for which maximization occurs.

This variational principle is not equivalent with the absolute minimization of Kähler action. Rather, HQ Universe would do its best to save energy, being as near as possible to vacuum. Also vacuum extremals would become physically relevant (they carry non-vanishing density gravitational energy). The non-determinism of the vacuum extremals would have an interpretation in terms of the ability of Universe to engineer itself. The attractiveness of the number theoretical variational principle from the point of calculability of TGD would be that the initial values for the time derivatives of the imbedding space coordinates at X^3 at light-like 7-D causal determinant could be computed by requiring that the energy of the solution is minimized. This could mean a computerizable construction of Kähler function.

If maximization occurs in $coHQ$ case, contrasts defined by the values of Kähler action in regions of definite sign of action density are maximized in $coHQ$ phase. The obvious objection is that this option de-stabilizes the theory by implying large fluctuations but one might argue that quantum criticality requires this. This option does not necessarily imply energy maximization. For instance, CP_2 type extremals are vacuum extremals. If $coHQ$ corresponds to a minimization for the dual of Kähler action in the sense described, situation changes. For instance, CP_2 type extremals correspond to a vanishing dual of Kähler action density. It must be emphasized that HQ and $coHQ$ phases need not be dual to each other but could correspond to disjoint regions of configuration space of 3-surfaces.

4.4 Generalizing the notion of $HO-H$ duality to quantum level

The obvious question is how the $HO-H$ duality could generalize to quantum level. Number theoretical considerations combined with the general vision about generalized Feynman diagrams as a generalization of braid diagrams lead to general formulas for vertices in HO picture.

Simple arguments lead to the conclusion that strict duality can make sense only if the hyper-octonionic spinor field is second quantized in some sense. One can imagine two, not necessarily mutually exclusive, manners to quantize.

a) The construction of the spinor structure for the configuration space of 3-surfaces in HO forces to concluded that HO spinor fields induced to $X^4 \subset HO$ are second quantized as usual and define configuration space gamma matrices as super generators. The classical real-analytic HO spinor fields would represent analogs of zero modes of H spinor fields. The second quantized part of hyper-octonionic spinor fields induced to $X^4 \subset HO$ would have $1+1+3+\overline{3}$ decomposition having interpretation in terms of quarks and leptons and second quantized oscillator operators would commute with hyper-octonionic units. The detailed realization of $HO-H$ duality suggests that the induced spinor fields at $X^4 \subset H$ *resp.* $X^4 \subset HO$ are restrictions of H *resp.* HO spinor fields. This would hold for zero modes and could hold for second quantized part too.

b) The original idea was that the real Laurent coefficients correspond to a complete set of mutually commuting Hermitian operators having interpretation as observables. This is not enough for configuration space geometry but is favored by quantum classical correspondence. Space-time concept would be well defined only for the eigen states of these operators and physical states are mapped to space-time surfaces. The Hermitian operators would naturally correspond to the state space spanned by super Kac-Moody and super-canonical algebras, and quantum states would have precise space-time counterparts in accordance with quantum-classical correspondence.

The regions inside which the power series representing HO analytic function and matrix elements of G_2 rotation converge are identified as counterparts of maximal deterministic regions of the space-time surface. The Hermitian operators replacing Laurent coefficients are assumed to commute inside these regions identifiable also as coherence regions for the generalized Schrödinger amplitude represented by the HO spinor field.

By quantum classical correspondence these regions would be correlates for the final states of quantum jumps. The space-like 3-D causal determinants X^3 bounding adjacent regions of this kind represent quantum jumps. The hyper-octonionic part of the inner of the hyper-octonionic spinor fields at the two sides of the discontinuity defined as an integral over X^3 would give a number identifiable as complex number when imaginary unit is identified appropriately. The inner product would be identified as a representation of S-matrix element for an internal transition of particle represented by 3-surface, that is 2-vertex.

For the generalized Feynman diagrams n-vertex corresponds to a fusion of n 4-surfaces along their ends at X^3. 3-vertex can be represented number theoretically as a triality of three hyper-octonion spinors integrated over the 3-surface in question. Higher vertices can be defined as composite functions of triality with a map $(h_1, h_2) \rightarrow \overline{h}_3$ defined by octonionic triality and by duality given by the inner product. More concretely, $m + n$ vertex corresponds in HO picture to the inner product for the local hyper-octonionic products of m outgoing and n incoming hyper-octonionic spinor fields integrated over the 3-surface defining the vertex. Both 2-vertices representing internal transitions and $n > 2$ vertices are completely fixed. This should give some idea about the power of the number theoretical vision.

One can raise objections against the need for non-conventional quantization. The number theoretic prescription does not apply to the second quantized parts of HO spinor fields and S-matrix elements can be constructed using them so that two equivalent prescriptions of S-matrix would emerge. On the other hand, TGD inspired quantum measurement theory suggests dual codings S-matrix elements based on either quantum states or classical observables (zero modes) in 1-1 correspondence with them.

4.4.1 Does TGD in HO picture reduce to 8-D WZW string model?

Conservation laws suggests that in the case of non-vacuum extremals the dynamics of the local G_2 automorphism is dictated by field equations of some kind. The experience with WZW model suggests that in the case of non-vacuum extremals G_2 element could be written as a product $g = g_L(h)g_R^{-1}(\overline{h})$ of hyper-octonion analytic and anti-analytic complexified G_2 elements. g would be determined by the data at hyper-complex 2-surface for which the tangent space at a given point is spanned by real unit and preferred hyper-octonionic unit. Also Dirac action would be naturally restricted to this surface. The theory would reduce in HO picture to 8-D WZW string model like theory both classically and quantally. The duality of partonic H description with stringy HO description suggests that string orbits correspond to surfaces at which time like causal determinants intersect.

The interpretation of generalized Feynman diagrams in terms of generalized braid/ribbon diagrams and the unique properties of G_2 provide further support for this picture. In particular, G_2 is the lowest-dimensional Lie group allowing to realize full-powered topological quantum computation based on generalized braid diagrams and using the lowest level k=1 Kac Moody representation. Even if this reduction would occur only in special cases, such as asymptotic solutions for which Lorentz Kähler force vanishes or maxima of Kähler function, it would mean enormous simplification of the theory.

4.4.2 Why hyper-quaternionicity corresponds to the minimization of Kähler action?

The resulting over all picture leads also to a considerable understanding concerning the basic questions why hyper-quaternionic 4-surfaces define extrema of Kähler action and why WZW strings would provide a dual for the description using Kähler action. The answer boils down to the realization that the extrema of Kähler action minimize complexity, also algebraic complexity, in particular non-commutativity. A measure for non-commutativity with a fixed preferred hyper-octonionic imaginary unit is provided by the commutator of 3 and $\overline{3}$ parts of the hyper-octonion spinor field defining an antisymmetric tensor in color octet representation: very much like color gauge field. Color action is a natural measure for the non-commutativity minimized when the tangent space algebra closes to complexified quaternionic, instead of complexified octonionic, algebra. On the other hand, Kähler action is nothing but color action for classical color gauge field defined by projections of color Killing vector fields. That WZW + Dirac action for hyper-octonionic strings would correspond to Kähler action would in turn be the TGD counterpart for the proposed string-YM dualities. If the dual of Kähler action defines $coHQ$ 4-surfaces an analogous interpretation holds true.

4.4.3 Various dualities and their physical counterparts

$HO-H$ duality is only one representative in a family of dualities characterizing TGD. It is not equivalent with $HQ-coHQ$ duality, which seems however to be equivalent with the electric-magnetic duality known for long. This duality relates descriptions based on space-like partonic 2-surfaces and time-like string orbits. $HO-H$ and $HQ-coHQ$ dualities seem to be closely correlated in the sense that HO picture is natural in HQ phase and H picture in $coHQ$ phase.

At configuration space level $HO-H$ duality means roughly following. In H picture spin and ew spin are spin-like quantum numbers whereas color is orbital quantum number and cannot be seen at space-time level directly. In HO picture the roles of these quantum numbers are changed. One could say that $HO-H$ duality acts as a super-symmetry permuting spin and orbital degrees of freedom of configuration space spinor fields. This duality allows a surprisingly detailed understanding of almost paradoxical dualities of hadron physics, and also explains proton spin crisis from first principles.

It seems possible to interpret $HO-H$ and $HQ-coHQ$ dualities as analogs of wave-particle duality in the infinite-dimensional context. For $HO-H$ duality the cotangent bundle of configuration space CH would be the unifying notion. Position q in CH would be represented by 3-surface whereas canonical momentum p would correspond to the same 3-surface but as a surface in CHO with induced metric and Kähler structure inherited from HO defining the tangent space of H. The notion of stringy configuration space and the generalization of HQ-coHQ duality to dimension-codimension duality might allow to understand also M-theory dualities in this manner.

5 TGD as a Generalized Number Theory: Infinite Primes

The notion of prime captures something very essential about what it is to be elementary building block of matter and has become a fundamental conceptual element of TGD. The notion of prime gains it generality from its reducibility to the notion of prime ideal of an algebra. Thus the notion of prime is well-defined, not only in case of quaternions and octonions, but also in the case of hyper-quaternions and -octonions, which are especially natural physically and for which numbers having zero norm correspond physically to light-like 8-vectors. In this chapter only infinite primes as generalization of ordinary primes are discussed and the interested reader can consult [E2] for the generalization of discussion to quaternionic and octonionic case which are most relevant for TGD.

5.1 The notion of infinite prime

p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump to some sector D_p of the configuration space labelled by a p-adic prime. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions representing selves with the corresponding decomposition of the infinite prime to primes at lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

This and other observations suggest that the Universe of quantum TGD might basically provide a physical representation of number theory allowing also infinite primes. The proposal suggests also a possible generalization of real numbers to a number system akin to hyper-reals introduced by Robinson in his non-standard calculus [7] providing rigorous mathematical basis for calculus. In fact, some rather natural requirements lead to a unique generalization for the concepts of integer, rational and real. Somewhat surprisingly, infinite integers and reals can be regarded as infinite-dimensional vector spaces with integer and real valued coefficients respectively and one cannot exclude the possibility that the tangent space for the configuration space of 3-surfaces could be regarded as the space of generalize 8-D hyper-octonionic numbers.

5.2 Generalization of ordinary number fields

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of hyper-octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

5.3 Infinite primes and physics in TGD Universe

Two different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

a) The first view which developed first, is based on the idea that infinite primes characterize quantum states of the entire Universe. 8-D hyper-octonions make this correspondence very concrete since 4-D hyper-quaternions have interpretation as four-momenta. By quantum-classical correspondence also the decomposition of space-time surfaces to space-time sheets characterized by p-adic primes. A further and very concrete aspect of this view is that infinite primes have a representation as 4-surfaces of 8-D hyper-octonionic imbedding space analogous to the curves of the complex plane defined by the vanishing of the imaginary part of a polynomial of a complex variable. Hence a generalization of algebraic geometry would emerge naturally and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity.

b) The second view is based on the idea that infinitely structured space-time points define space-time correlates of mathematical cognition. The mathematical analog of Brahman=Atman identity would however suggest that both views deserve to be taken seriously.

5.3.1 Infinite primes and infinite hierarchy of second quantizations

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. Later it became clear that the process generalizes so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts This hierarchy of second quantizations means enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. The representations of color group $SU(3)$ are indeed labelled by two integers and the states inside given representation by color hypercharge and iso-spin.

5.3.2 Infinite primes as a bridge between quantum and classical

An important stimulus came from the observation stimulated by algebraic number theory [1]. Infinite primes can be mapped to polynomial primes and this observation allows to identify completely generally the spectrum of infinite primes whereas hitherto it was possible to construct explicitly only what might be called generating infinite primes.

This in turn led to the observation that one can represent infinite primes (integers) geometrically as surfaces associated with the polynomials associated with infinite primes (integers). Not surprisingly, the first attempts to guess this relationship turned out to be too naive. The recent proposal is based on the number theoretic construction of solutions of field equations carried out in the previous chapter [E2]. Therefore algebraic geometric description of the space-time surfaces suggests itself, and the natural question is whether it provides a general solution to field equations.

Infinite primes would serve as a bridge between Fock-space descriptions and geometric descriptions of physics: quantum and classical. Geometric objects could be seen as concrete representations of infinite numbers providing amplification of infinitesimals to macroscopic deformations of space-time surface. We see the infinitesimals as concrete geometric shapes!

5.3.3 Various equivalent characterizations of space-times as surfaces

One can imagine several number-theoretic characterizations of the space-time surface.

a) The approach based on octonions and quaternions suggests that space-time surfaces might correspond to associative or hyper-quaternionic surfaces of hyper-octonionic imbedding space. The meaning of "associative" depends on how one induces the octonion algebra to the space-time surface. Also the identification of space-time surfaces as 4-surfaces for which tangent space at each point is algebraically closed in some appropriate sense, can be considered. Also the normal space of X^4 could have the property in question. Hyper-quaternionicity or its co-counterpart seems to be the correct property and implies associativity and commutativity when induced algebra structure is defined by dropping from the product the part which is normal to the tangent space of X^4.

b) Space-time surfaces could be seen as an absolute minima of the Kähler action. The great challenge is to rigorously prove that this characterization is equivalent with the others.

5.3.4 The representation of infinite primes as 4-surfaces

The original idea was that space-time surfaces could be regarded as four-surfaces X^4 of 8-D imbedding space $H = M^4 \times CP_2$ having the property that the tangent space of X^4 *resp.* H can be locally regarded as 4- *resp.* 8-dimensional field of quaternions *resp.* octonions. The difficulties caused by the Euclidian metric signature of the number theoretical norm however forced to give up the idea in its original form, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart for the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The notions of hyper-quaternionic and octonionic manifold make sense but it is implausible that $H = M^4 \times CP_2$ could be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces are assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with a fixed complex structure are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

The construction of general solutions of field equations based on this idea is discussed in the previous chapter [E2]. Any hyper-octonion analytic function $OH \rightarrow OH$ defines a function $g : OH \rightarrow SU(3)$ acting as the group of octonion automorphisms leaving selected imaginary unit invariant, and g in turn defines a foliation of OH and $H = M^4 \times CP_2$ by space-time surfaces. The selection can be local which means that G_2 appears as a local gauge group.

Since the notion of prime makes sense for the complexified octonions, it makes sense also for the hyper-octonions. It is possible to assign to infinite prime of this kind a hyper-octonion analytic polynomial $P : OH \rightarrow OH$ and hence also a foliation of OH and $H = M^4 \times CP_2$ by 4-surfaces. Therefore space-time surface can be seen as a geometric counterpart of a Fock state. The assignment is not unique but determined only up to an element of the local octonionic automorphism group G_2 acting in HO and fixing the local choices of the preferred imaginary unit of the hyper-octonionic tangent plane. In fact, a map $HO \rightarrow S^6$ characterizes the choice since $SO(6)$ acts effectively as a local gauge group.

The construction generalizes to all levels of the hierarchy of infinite primes and produces also representations for integers and rationals associated with hyper-octonionic numbers as space-time surfaces. A close relationship with algebraic geometry results and the polynomials define a natural hierarchical structure in the space of 3-surfaces. By the effective 2-dimensionality naturally associated with infinite primes represented by real polynomials 4-surfaces are determined by data given at partonic 2-surfaces defined by the intersections of 3-D and 7-D light-like causal determinants. In particular, the notions of genus and degree serve as classifiers of the algebraic geometry of the 4-surfaces.

5.3.5 Infinite primes and quantum gravitational holography

Infinite primes emerge naturally in the realization of the quantum gravitational holography in terms of the modified Dirac operator and provide a deeper understanding of the basic aspects of the configuration space geometry.

a) Two types of infinite primes are predicted corresponding to the two types of fermionic vacua $X \pm 1$, where X is the product of all finite primes. The physical interpretation for the two types of infinite primes $X \pm 1$ is in terms of two quantizations for which creation and oscillator operators change role and which correspond to the two signs of inertial energy in TGD Universe. In particular, phase conjugate photons would be negative energy photons erratically believed to reduce to standard physics.

b) The new view about gravitational and inertial masses forced by TGD leads also the view that positive and negative energy space-time sheets are created pairwise at space-like 3-surfaces located at 7-D light-like causal determinants $X^7_\pm = \delta M^4_\pm \times CP_2$. The conjecture is that the ratio of Dirac determinants associated with the positive and negative energy space-time sheets, which is finite, equals to the exponent of Kähler function which would be thus determined completely by the data at 3-dimensional causal determinants and realizing quantum gravitational holography.

c) The spectra associated with the space-time sheets X^4_+ and X^4_- meeting at X^3 would correspond to the infinite primes built from the vacua corresponding to the infinite primes $X \pm 1$. The close analogy of the product of all finite hyper-octonionic primes with Dirac determinant suggest that the ratio of the determinants corresponds to the ratio of infinite primes defining X^4_+ and X^4_-. The theory predicts the dependence of the eigenvalues of the modified Dirac operator on the value of the Kähler action. Both Kähler coupling strength and gravitational coupling strength are expressible in terms of the finite primes characterizing the ratio of the infinite primes and this ratio depends on the p-adic prime characterizing X^4_+ and X^4_-.

d) Some modes of the spectrum of the modified Dirac operator at $X^4_\pm$ become zero modes, and by the resulting spectral asymmetry the ratio of the determinants differs from unity. Thus the spectral asymmetry or the infinite primes defining the space-time sheets X^4_+ and X^4_- is all that would be needed to deduce the value of the vacuum functional once causal determinants are known.

5.4 Complete algebraic, dimensional, and topological democracy?

Without the notion of Platonia allowing realization of all imaginable algebraic structures cognitively but leaving no trace on the physics of matter, the idea about dimensional democracy would look almost compelling despite the fact that it might well be in conflict with the special role of the dimensions associated with the classical number fields. One can imagine several realizations of this idea.

a) The most (if not the only) plausible realization for the dimensional hierarchy would be following. Both fractal cosmology, non-determinism of Kähler action, and Poincare invariance favor the option in which configuration space is a union of sectors characterized by unions of future and past light cones $M^4_\pm(a)$ where a characterizes the position a of the dip of the light-cone in M^4. Future/past dichotomy would correspond to positive/negative energy dichotomy and to the two kinds of infinite primes constructed from

$X \pm 1$, X the product of all finite primes. Hence the cm degrees of freedom for the sectors of the configuration space would correspond to the union of the spaces $(M^4)^m \times (M^4)^n$ of dimension $D = 4(m+n)$, and the dimensional democracy would conform with the 8-dimensionality of the imbedding space.

b) The most plausible identification consistent with the p-adic length scale hierarchy is as unions of n disjoint 4-surfaces of H. This correspondence is completely analogous to that involved when the configuration space of n point-like particles is identified as $(E^3)^n$ in wave mechanics.

c) One might also consider of assigning with hyper-octonionic infinite primes of level n $4n$-dimensional surfaces in $8n$-dimensional space $H^n = (M^4_+ \times CP_2)^n$. This would suggests a dimensional hierarchy of space-time surfaces and a complete dimensional and algebraic democracy: quite a considerable generalization of quantum TGD from its original formulation. This option does not however look physically plausible since it is not consistent with the hierarchical "abstractions about abstractions" structure of infinite primes and corresponding space-time representations.

Since quantum field theories are based on the notion of point like particles, the hierarchy of arithmetic quantum field theories associated with infinite primes cannot code entire quantum TGD but only the ground states of the super-canonical representations. This might however be the crucial element needed to understand the construction S-matrix of quantum TGD at the general level.

One can imagine also a topological democracy and an evolution of algebraic topological structures. At the lowest, primordial level there are just algebraic surfaces allowing no completion to smooth ...-adic or real surfaces, and defined only in algebraic extensions of rationals by algebraic field equations. At higher levels rational-adic, p-adic and even infinite-P p-adic completions of infinite primes could appear and provide natural completions of function spaces. Of course, all these generalizations might make sense only as cognitive structures in Platonia and it is comforting to know that there is room in just a single point of TGD Universe for all this richness of imaginable structures!

The reader not familiar with the basic algebra of quaternions and octonions is encourated to study some background material: the homepage of Tony Smith provides among other things an excellent introduction to quaternions and octonions [2]. String model builders are beginning to grasp the potential importance of octonions and quaternions and the articles about possible applications of octonions [3, 4, 5] provide an introduction to octonions using the language of physicist.

Personally I found quite frustrating to realize that I had neglected totally learning of the basic ideas of algebraic geometry, despite its obvious potential importance for TGD and its applications in string models. This kind of losses are the price one must pay for working outside the scientific community. It is not easy for a physicist to find readable texts about algebraic geometry and algebraic number theory from the bookshelves of mathematical libraries. The book "Algebraic Geometry for Scientists and Engineers" by Abhyankar [6], which is not so elementary as the name would suggest, introduces in enjoyable manner the basic concepts of algebraic geometry and binds the basic ideas with the more recent developments in the field. "Problems in Algebraic Number Theory" by Esmonde and Murty [1] in turn teaches algebraic number theory through exercises which concretize the abstract ideas. The book "Invitation to Algebraic Geometry" by K. E. Smith. L. Kahanpää, P. Kekäläinen and W. Traves is perhaps the easiest and most enjoyable introduction to the topic for a novice. It also contains references to the latest physics inspired work in the field.

6 Infinite primes, integers, and rationals

By the arguments of introduction p-adic evolution leads to a gradual increase of the p-adic prime p and at the limit $p \to \infty$ Omega Point is reached in the sense that the negentropy gain associated with quantum jump can become arbitrarily large. There several interesting questions to be answered. Does the topology R_P at the limit of infinite P indeed approximate real topology? Is it possible to generalize the concept of prime number and p-adic number field to include infinite primes? This is is possible is suggested by the fact that sheets of 3-surface are expected to have infinite size and thus to correspond to infinite p-adic length scale. Do p-adic numbers R_P for sufficiently large P give rise to reals by canonical identification? Do the number fields R_P provide an alternative formulation/generalization of the non-standard analysis based on the hyper-real numbers of Robinson [7]? Is it possible to generalize the adelic formula [E4] so that one could generalize quantum TGD so that it allows effective p-adic topology for infinite values of p-adic prime? It must be emphasized that the consideration of infinite primes need not be a purely academic exercise: for infinite values of p p-adic perturbation series contains only two terms and this limit, when properly formulated, could give excellent approximation of the finite p theory for large p.

It turns out that there is not any unique infinite prime nor even smallest infinite prime and that there is an entire hierarchy of infinite primes. Somewhat surprisingly, R_P is not mapped to entire set of reals nor even rationals in canonical identification: the image however forms a dense subset of reals. Furthermore, by introducing the corresponding p-adic number fields R_P, one necessarily obtains something more than reals: one might hope that for sufficiently large infinite values of P this something might be regarded as a generalization of real numbers to a number field containing both infinite numbers and infinitesimals.

The pleasant surprise is that one can find a general construction recipe for infinite primes and that this recipe can be characterized as a repeated second quantization procedure in which the many boson states of the previous level become single boson states of the next level of the hierarchy: this realizes Cantor's definition 'Set as Many allowing to regard itself as One' in terms of the basic concepts of quantum physics. Infinite prime allows decomposition to primes at lower level of infinity and these primes can be identified as primes labelling various space-time sheets which are in turn geometric correlates of selves in TGD inspired theory of consciousness. Furthermore, each infinite prime defines decomposition of a fictive many particle state to a purely bosonic part and to part for which fermion number is one in each mode. This decomposition corresponds to the decomposition of the space-time surface to cognitive and material space-time sheets. Thus the concept of infinite prime suggests completely unexpected connection between quantum field theory, TGD based theory of consciousness and number theory by providing in its structure nothing but a symbolic representation of mathematician and external world!

The definition of the infinite integers and rationals is a straightforward procedure. Infinite primes also allow generalization of the notion of ordinary number by allowing infinite-dimensional space of real units which are however non-equivalent in p-adic sense. This means that space-time points are infinitely structured. The fact that this structure completely invisible at the level of real physics suggests that it represents the space-time correlate for mathematical cognition.

6.1 The first level of hierarchy

In the following the concept of infinite prime is developed gradually by stepwise procedure rather than giving directly the basic definitions. The hope is that the development of the concept in the same manner as it actually occurred would make it easier to understand it.

Step 1

One could try to define infinite primes P by starting from the basic idea in the proof of Euclid for the existence of infinite number of primes. Take the product of all finite primes and add 1 to get a new prime:

$$\begin{aligned} P &= 1 + X \ , \\ X &= \prod_p p \ . \end{aligned} \tag{5}$$

If P were divisible by finite prime then $P - X = 1$ would be divisible by finite prime and one would encounter contradiction. One could of course worry about the possible existence of infinite primes smaller than P and possibly dividing P. The numbers $N = P - k$, $k > 1$, are certainly not primes since k can be taken as a factor. The number $P' = P - 2 = -1 + X$ could however be prime. P is certainly not divisible by $P - 2$. It seems that one cannot express P and $P - 2$ as product of infinite integer and finite integer. Neither it seems possible to express these numbers as products of more general numbers of form $\prod_{p \in U} p + q$, where U is infinite subset of finite primes and q is finite integer.

Step 2

P and $P - 2$ are not the only possible candidates for infinite primes. Numbers of form

$$\begin{aligned} P(\pm, n) &= \pm 1 + nX \ , \\ k(p) &= 0, 1, ... \ , \\ n &= \prod_p p^{k(p)} \ , \\ X &= \prod_p p \ , \end{aligned} \tag{6}$$

where $k(p) \neq 0$ holds true only in finite set of primes, are characterized by a integer n, and are also good prime candidates. The ratio of these primes to the prime candidate P is given by integer n. In general, the ratio of two prime candidates $P(m)$ and $P(n)$ is rational number m/n telling which of the prime candidates is larger. This number provides ordering of the prime candidates $P(n)$. The reason why these numbers are good canditates for infinite primes is the same as above. No finite prime p with $k(p) \neq 0$ appearing in the product can divide these numbers since, by the same arguments as appearing in Euclid's theorem, it would divide also 1. On the other hand it seems difficult to invent any decomposition of these numbers containing infinite numbers. Already at this stage one can notice the structural analogy with the construction of multiboson states in quantum field theory: the numbers $k(p)$ correspond to the occupation numbers of bosonic states of quantum field theory in one-dimensional box, which suggests that the basic structure of QFT might have number theoretic interpretation in some very general sense. It turns out that this analogy generalizes.

Step 3

All $P(n)$ satisfy $P(n) \geq P(1)$. One can however also the possibility that $P(1)$ is not the smallest infinite prime and consider even more general candidates for infinite primes, which are smaller than $P(1)$. The trick is to drop from the infinite product of primes $X = \prod_p p$ some primes away by dividing it by integer $s = \prod_{p_i} p_i$, multiply this number by an integer n not divisible by any prime dividing s and to add to/subtract from the resulting number nX/s natural number ms such that m expressible as a product of powers of only those primes which appear in s to get

$$\begin{aligned} &P(\pm, m, n, s) = n\frac{X}{s} \pm ms \ , \\ &m = \prod_{p|s} p^{k(p)} \ , \\ &n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{aligned} \tag{7}$$

Here $x|y$ means 'x divides y'. To see that no prime p can divide this prime candidate it is enough to calculate $P(\pm, m, n, s)$ modulo p: depending on whether p divides s or not, the prime divides only the second term in the sum and the result is nonzero and finite (although its precise value is not known). The ratio of these prime candidates to $P(+, 1, 1, 1)$ is given by the rational number n/s: the ratio does not depend on the value of the integer m. One can however order the prime candidates with given values of n and s using the difference of two prime candidates as ordering criterion. Therefore these primes can be ordered.

One could ask whether also more general numbers of the form $n\frac{X}{s} \pm m$ are primes. In this case one cannot prove the indivisibility of the prime candidate by p not appearing in m. Furthermore, for $s \ mod \ 2 = 0$ and $m \ mod \ 2 \neq 0$, the resulting prime candidate would be even integer so that it looks improbable that one could obtain primes in more general case either.

Step 4

An even more general series of candidates for infinite primes is obtained by using the following ansatz which in principle is contained in the original ansatz allowing infinite values of n

$$
\begin{aligned}
&P(\pm, m, n, s|r) = nY^r \pm ms \ , \\
&Y = \tfrac{X}{s} \ , \\
&m = \textstyle\prod_{p|s} p^{k(p)} \ , \\
&n = \textstyle\prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ .
\end{aligned} \tag{8}
$$

The proof that this number is not divisible by any finite prime is identical to that used in the previous case. It is not however clear whether the ansatz for given r is not divisible by infinite primes belonging to the lower level. A good example in $r = 2$ case is provided by the following unsuccessful ansatz

$$
\begin{aligned}
&N = (n_1Y + m_1s)(n_2Y + m_2s) = \tfrac{n_1n_2X^2}{s^2} - m_1m_2s^2 \ , \\
&Y = \tfrac{X}{s} \ , \\
&n_1m_2 - n_2m_1 = 0 \ .
\end{aligned}
$$

Note that the condition states that n_1/m_1 and $-n_2/m_2$ correspond to the same rational number or equivalently that (n_1, m_1) and (n_2, m_2) are linearly dependent as vectors. This encourages the guess that all other $r = 2$ prime candidates with finite values of n and m at least, are primes. For higher values of r one can deduce analogous conditions guaranteing that the ansatz does not reduce to a product of infinite primes having smaller value of r. In fact, the conditions for primality state that the polynomial $P(n, m, r)(Y) = nY^r + m$ with integer valued coefficients ($n > 0$) defined by the prime candidate is irreducible in the field of integers, which means that it does not reduce to a product of lower order polynomials of same type.

Step 5

A further generalization of this ansatz is obtained by allowing infinite values for m, which leads to the following ansatz:

$$
\begin{aligned}
&P(\pm, m, n, s|r_1, r_2) = nY^{r_1} \pm ms \ , \\
&m = P_{r_2}(Y)Y + m_0 \ , \\
&Y = \tfrac{X}{s} \ , \\
&m_0 = \textstyle\prod_{p|s} p^{k(p)} \ , \\
&n = \textstyle\prod_{p|Y} p^{k(p)}, \ \ k(p) \geq 0 \ .
\end{aligned} \tag{9}
$$

Here the polynomial $P_{r_2}(Y)$ has order r_2 is divisible by the primes belonging to the complement of s so that only the finite part m_0 of m is relevant for the divisibility by finite primes. Note that the part proportional to s can be infinite as compared to the part proportional to Y^{r_1}: in this case one must however be careful with the signs to get the sign of the infinite prime correctly. By using same arguments as earlier one finds that these prime candidates are not divisible by finite primes. One must also require that the ansatz is not divisible by lower order infinite primes of the same type. These conditions are equivalent to the conditions guaranteing the polynomial primeness for polynomials of form $P(Y) = nY^{r_1} \pm (P_{r_2}(Y)Y + m_0)s$ having integer-valued coefficients. The construction of these polynomials can be performed recursively by starting from the first order polynomials representing first level infinite primes: Y can be regarded as formal variable and one can forget that it is actually infinite number.

By finite-dimensional analogy, the infinite value of m means infinite occupation numbers for the modes represented by integer s in some sense. For finite values of m one can always write m as a product of powers of $p_i|s$. Introducing explicitly infinite powers of p_i is not in accordance with the idea that all exponents appearing in the formulas are finite and that the only infinite variables are X and possibly S (formulas are symmetric with respect to S and X/S). The proposed representation of m circumvents this difficulty in an elegant manner and allows to say that m is expressible as a product of infinite powers of p_i despite the fact that it is not possible to derive the infinite values of the exponents of p_i.

Summarizing, an infinite series of candidates for infinite primes has been found. The prime candidates $P(\pm, m, n, s)$ labelled by rational numbers n/s and integers m plus the primes $P(\pm, m, n, s|r_1, r_2)$ constructed as r_1:th or r_2:th order polynomials of $Y = X/s$: the latter ansatz reduces to the less general ansatz of infinite values of n are allowed.

One can ask whether the $p \ mod \ 4 = 3$ condition guaranteing that the square root of -1 does not exist as a p-adic number, is satisfied for $P(\pm, m, n, s)$. $P(\pm, 1, 1, 1) \ mod \ 4$ is either 3 or 1. The value of $P(\pm, m, n, s) \ mod \ 4$ for odd s on n only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. For even s the value of $P(\pm, m, n, s) \ mod \ 4$ depends on m only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. This condition resembles G-parity condition of Super Virasoro algebras. Note that either $P(+, m, n, s)$ or $P(-, m, n, s)$ but not both are physically interesting infinite primes ($2m \ mod \ 4 = 2$ for odd m) in the sense of allowing complex Hilbert space. Also the additional conditions satisfied by the states involving higher powers of X/s resemble to Virasoro conditions. An open problem is whether the analogy with the construction of the many-particle states in super-symmetric theory might be a hint about more deeper relationship with the representation of Super Virasoro algebras and related algebras.

It is not clear whether even more general prime candidates exist. An attractive hypothesis is that one could write explicit formulas for all infinite primes so that generalized theory of primes would reduce to the theory of finite primes.

6.2 Infinite primes form a hierarchy

By generalizing using general construction recipe, one can introduce the second level prime candidates as primes not divisible by any finite prime p or infinite prime candidate of type $P(\pm, m, n, s)$ (or more general prime at the first level: in the following we assume for simplicity that these are the only infinite primes at the first level). The general form of these prime candidates is exactly the same as at the first level. Particle-analogy makes it easy to express the construction receipe. In present case 'vacuum primes' at the lowest level are of the form

$$\begin{array}{l} \frac{X_1}{S} \pm S \ , \\ X_1 = X \prod_{P(\pm,m,n,s)} P(\pm, m, n, s) \ , \\ S = s \prod_{P_i} P_i \ , \\ s = \prod_{p_i} p_i \ . \end{array} \tag{10}$$

S is product or ordinary primes p and infinite primes $P_i(\pm, m, n, s)$. Primes correspond to physical states created by multiplying X_1/S (S) by integers not divisible by primes appearing S (X_1/S). The integer valued functions $k(p)$ and $K(p)$ of prime argument give the occupation numbers associated with X/s and s type 'bosons' respectively. The non-negative integer-valued function $K(P) = K(\pm, m, n, s)$ gives the occupation numbers associated with the infinite primes associated with X_1/S and S type 'bosons'. More general primes can be constructed by mimicking the previous procedure.

One can classify these primes by the value of the integer $K_{tot} = \sum_{P|X/S} K(P)$: for a given value of K_{tot} the ratio of these prime candidates is clearly finite and given by a rational number. At given level the ratio P_1/P_2 of two primes is given by the expression

$$\frac{P_1(\pm,m_1,n_1,s_1 K_1,S_1}{P_2(\pm,m_2,n_2,s_2,K,S_2)} = \frac{n_1 s_2}{n_2 s_1} \prod_{\pm,m,n,s} (\frac{n}{s})^{K_1^+(\pm,n,m,s) - K_2^+(\pm,n,m,s)} \ . \tag{11}$$

Here K_i^+ denotes the restriction of $K_i(P)$ to the set of primes dividing X/S. This ratio must be smaller than 1 if it is to appear as the first order term $P_1 P_2 \rightarrow P_1/P_2$ in the canonical identification and again it seems that it is not possible to get all rationals for a fixed value of P_2 unless one allows infinite values of N expressed neatly using the more general ansatz involving higher power of S.

6.3 Construction of infinite primes as a repeated quantization of a super-symmetric arithmetic quantum field theory

The procedure for constructing infinite primes is very much reminiscent of the second quantization of an super-symetric arithmetic quantum field theory in which single particle fermion and boson states are labelled by primes. In particular, there is nothing especially frightening in the particle representation of infinite primes: theoretical physicists actally use these kind of representations quite routinely.

a) The binary-valued function telling whether a given prime divides s can be interpreted as a fermion number associated with the fermion mode labelled by p. Therefore infinite prime is characterized by bosonic and fermionic occupation numbers as functions of the prime labelling various modes and situation is super-symmetric. X can be interpreted as the counterpart of Dirac sea in which every negative energy state state is occupied and $X/s \pm s$ corresponds to the state containing fermions understood as holes of Dirac sea associated with the modes labelled by primes dividing s.

b) The multiplication of the 'vacuum' X/s with $n = \prod_{p|X/s} p^{k(p)}$ creates $k(p)$ 'p-bosons' in mode of type X/s and multiplication of the 'vacuum' s with $m = \prod_{p|s} p^{k(p)}$ creates $k(p)$ 'p-bosons'. in mode of type s (mode occupied by fermion). The vacuum states in which bosonic creation operators act, are tensor products of two vacuums with tensor product represented as sum

$$|vac(\pm)\rangle = |vac(\frac{X}{s})\rangle \otimes |vac(\pm s)\rangle \leftrightarrow \frac{X}{s} \pm s \qquad (12)$$

obtained by shifting the prime powers dividing s from the vacuum $|vac(X)\rangle = X$ to the vacuum ± 1. One can also interpret various vacuums as many fermion states. Prime property follows directly from the fact that any prime of the previous level divides either the first or second factor in the decomposition $NX/S \pm MS$.

c) This picture applies at each level of infinity. At a given level of hierarchy primes P correspond to all the Fock state basis of all possible many-particle states of second quantized super-symmetric theory. At the next level these many-particle states are regarded as single particle states and further second quantization is performed so that the primes become analogous to the momentum labels characterizing various single-particle states at the new level of hierarchy.

d) There are two nonequivalent quantizations for each value of S due to the presence of $\pm$ sign factor. Two primes differing only by sign factor are like G-parity $+1$ and -1 states in the sense that these primes satisfy $P\ mod\ 4 = 3$ and $P\ mod\ 4 = 1$ respectively. The requirement that -1 does not have p-adic square root so that Hilbert space is complex, fixes G-parity to say $+1$. This observation suggests that there exists a close analogy with the theory of Super Virasoro algebras so that quantum TGD might have interpretation as number theory in infinite context. An alternative interpretation for the $\pm$ degeneracy is as counterpart for the possibility to choose the fermionic vacuum to be a state in which either all positive or all negative energy fermion states are occupied.

e) One can also generalize the construction to include polynomials of $Y = X/S$ to get infinite hierarchy of primes labelled by the two integers r_1 and r_2 associated with the polynomials in question. An entire hierarchy of vacuums labelled by r_1 is obtained. A possible interpretation of these primes is as counterparts for the bound states of quantum field theory. The coefficient for the power $(X/s)^{r_1}$ appearing in the highest term of the general ansatz, codes the occupation numbers associated with vacuum $(X/s)^{r_1}$. All the remaining terms are proportional to s and combine to form, in general infinite, integer m characterizing various infinite occupation numbers for the subsystem characterized by s. The additional conditions guaranteeing prime number property are equivalent with the primality conditions for polynomials with integer valued coefficients and resemble Super Virasoro conditions. For $r_2 > 0$ bosonic occupation numbers associated with the modes with fermion number one are infinite and one cannot write explicit formula for the boson number.

f) One could argue that the analogy with super-symmetry is not complete. The modes of Super Virasoro algebra are labelled by natural number whereas now modes are labelled by prime. This need not be a problem since one can label primes using natural number n. Also 8-valued spin index associated with fermionic and bosonic single particle states in TGD world is lacking (space-time is surface in 8-dimensional space). This index labels the spin states of 8-dimensional spinor with fixed chirality. One could perhaps get also spin index by considering infinite octonionic primes, which correspond to vectors of 8-dimensional integer lattice such that the length squared of the lattice vector is ordinary prime:

$$\sum_{k=1,...,8} n_k^2 = \text{ prime } .$$

Thus one cannot exclude the possibility that TGD based physics might provide representation for octonions extended to include infinitely large octonions. The notion of prime octonion is well defined in the set of integer octonions and it is easy to show that the Euclidian norm squared for a prime octonion is prime. If this result generalizes then the construction of generalized prime octonions would generalize the construction of finite prime octonions. It would be interesting to know whether the results of finite-dimensional case might generalize to the infinite-dimensional context. One cannot exclude the possibility that prime octonions are in one-one correspondence with physical states in quantum TGD.

These observations suggest a close relationship between quantum TGD and the theory of infinite primes in some sense: even more, entire number theory and mathematics might be reducible to quantum physics understood properly or equivalently, physics might provide the representation of basic mathematics. Of course, already the uniqueness of the basic mathematical structure of quantum TGD points to this direction. Against this background the fact that 8-dimensionality of the imbedding space allows introduction of octonion structure (also p-adic algebraic extensions) acquires new meaning. Same is also suggested by the fact that the algebraic extensions of p-adic numbers allowing square root of real p-adic number are 4- and 8-dimensional.

What is especially interesting is that the core of number theory would be concentrated in finite primes since infinite primes are obtained by straightforward procedure providing explicit formulas for them. Repeated quantization provides also a model of abstraction process understood as construction of hierarchy of natural number valued functions about functions about ...... At the first level infinite primes are characterized by the integer valued function $k(p)$ giving occupation numbers plus subsystem-complement division (division to thinker and external world!). At the next level prime is characterized in a similar manner. One should also notice that infinite prime at given level is characterized by a pair $(R = MN, S)$ of integers at previous level. Equivalently, infinite prime at given level is characterized by fermionic and bosonic occupation numbers as functions in the set of primes at previous level.

6.4 Construction in the case of an arbitrary commutative number field

The basic construction recipe for infinite primes is simple and generalizes even to the case of algebraic extensions of rationals. Let $K = Q(\theta)$ be an algebraic number field (see the Appendix of [E1] for the basic definitions). In the general case the notion of prime must be replaced by the concept of irreducible defined as an algebraic integer with the property that all its decompositions to a product of two integers are such that second integer is always a unit (integer having unit algebraic norm, see Appendix of [E1]).

Assume that the irreducibles of $K = Q(\theta)$ are known. Define two irreducibles to be equivalent if they are related by a multiplication with a unit of K. Take one representative from each equivalence class of units. Define the irreducible to be positive if its first non-vanishing component in an ordered basis for the algebraic extension provided by the real unit and powers of θ, is positive. Form the counterpart of Fock vacuum as the product X of these representative irreducibles of K.

The unique factorization domain (UFD) property (see Appendix of [E1]) of infinite primes does not require the ring O_K of algebraic integers of K to be UFD although this property might be forced somehow. What is needed is to find the primes of K; to construct X as the product of all irreducibles of K but not counting units which are integers of K with unit norm; and to apply second quantization to get primes which are first order monomials. X is in general a product of powers of primes. Generating infinite primes at the first level correspond to generalized rationals for K having similar representation in terms of powers of primes as ordinary rational numbers using ordinary primes.

6.5 Mapping of infinite primes to polynomials and geometric objects

The mapping of the generating infinite primes to first order monomials labelled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \rightarrow x_{\pm} \pm \frac{m}{sn} \quad . \tag{13}$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has rational coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n:th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n:th level one has polynomials $P(q_1|q_2|...)$ of q_1 with coefficients which are rational functions of q_2 with coefficients which are.... The hierarchy of infinite primes is thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on .... At second levels the representation of infinite primes is an algebraic curve resulting as a locus of $P(q_1|q_2) = 0$. At higher levels the locus is a higher dimensional surface.

The replacement of rational numbers with rationals of higher-dimensional number field or algebra of hyper-octonions is possible and the notion of prime makes sense also in this case. In this case infinite primes allow a geometric representation as space-time surfaces as hyper-quaternionic sub-manifolds. The construction of representation is however more intricate than in case of algebraic curves.

6.6 How to order infinite primes?

One can order the infinite primes, integers and rationals. The ordering principle is simple: one can decompose infinite integers to two parts: the 'large' and the 'small' part such that the ratio of the small part with the large part vanishes. If the ratio of the large parts of two infinite integers is different from one or their sign is different, ordering is obvious. If the ratio of the large parts equals to one, one can perform same comparison for the small parts. This procedure can be continued indefinitely.

In case of infinite primes ordering procedure goes like follows. At given level the ratios are rational numbers. There exists infinite number of primes with ratio 1 at given level, namely the primes with same values of N and same S with MS infinitesimal as compared to NX/S. One can order these primes using either the relative sign or the ratio of $(M_1S_1)/(M_2S_2)$ of the small parts to decide which of the two is larger. If also this ratio equals to one, one can repeat the process for the small parts of M_iS_i. In principle one can repeat this process so many times that one can decide which of the two primes is larger. Same of course applies to infinite integers and also to infinite rationals build from primes with infinitesimal MS. If NS is not infinitesimal it is not obvious whether this procedure works. If $N_iX_i/M_iS_i = x_i$ is finite for both numbers (this need not be the case in general) then the ratio $\frac{M_1S_1}{M_2S_2}\frac{(1+x_2)}{(1+x_1)}$ provides the needed criterion. In case that this ratio equals one, one can consider use the ratio of the small parts multiplied by $\frac{(1+x_2)}{(1+x_1)}$ of M_iS_i as ordering criterion. Again the procedure can be repeated if needed.

6.7 What is the cardinality of infinite primes at given level?

The basic problem is to decide whether Nature allows also integers S , $R = MN$ represented as infinite product of primes or not. Infinite products correspond to subsystems of infinite size (S) and infinite total occupation number (R) in QFT analogy.

a) One could argue that S should be a finite product of integers since it corresponds to the requirement of finite size for a physically acceptable subsystem. One could apply similar argument to R. In this case the set of primes at given level has the cardinality of integers ($alef_0$) and the cardinality of all infinite primes is that of integers. If also infinite integers R are assumed to involve only finite products of infinite primes the set of infinite integers is same as that for natural numbers.

b) NMP is well defined in p-adic context also for infinite subsystems and this suggests that one should allow also infinite number of factors for both S and $R = MN$. Super symmetric analogy suggests the same: one can quite well consider the possibility that the total fermion number of the universe is infinite. It seems however natural to assume that the occupation numbers $K(P)$ associated with various primes P in the representations $R = \prod_P P^{K(P)}$ are finite but nonzero for infinite number of primes P. This requirement applied to the modes associated with S would require the integer m to be explicitly expressible in powers of $P_i|S$ ($P_{r_2} = 0$) whereas all values of r_1 are possible. If infinite number of prime factors is allowed in the definition of S, then the application of diagonal argument of Cantor shows that the number of infinite primes is larger than $alef_0$ already at the first level. The cardinality of the first level is $2^{alef_0}2^{alef_0} == 2^{alef_0}$. The first factor is the cardinality of reals and comes from the fact that the sets S form the set of all possible subsets of primes, or equivalently the cardinality of all possible binary valued functions in the set of primes. The second factor comes from the fact that integers $R = NM$ (possibly infinite) correspond to all natural number-valued functions in the set of primes: if only finite powers $k(p)$ are allowed then one can map the space of these functions to the space of binary valued functions bijectively and the cardinality must be 2^{alef_0}. The general formula for the cardinality at given level is obvious: for instance, at the second level the cardinality is the cardinality of all possible subsets of reals. More generally, the cardinality for a given level is the cardinality for the subset of possible subsets of primes at the previous level.

6.8 How to generalize the concepts of infinite integer, rational and real?

The allowance of infinite primes forces to generalize also the concepts concepts of integer, rational and real number. It is not obvious how this could be achieved. The following arguments lead to a possible generalization which seems practical (yes!) and elegant.

6.8.1 Infinite integers form infinite-dimensional vector space with integer coefficients

The first guess is that infinite integers N could be defined as products of the powers of finite and infinite primes.

$$N = \prod_k p_k^{n_k} = nM \ , \ n_k \geq 0 \ , \tag{14}$$

where n is finite integer and M is infinite integer containing only powers of infinite primes in its product expansion.

It is not however not clear whether the sums of infinite integers really allow similar decomposition. Even in the case that this decomposition exists, there seems to be no way of deriving it. This would suggest that one should regard sums

$$\sum_i n_i M_i$$

of infinite integers as infinite-dimensional linear space spanned by M_i so that the set of infinite integers would be analogous to an infinite-dimensional algebraic extension of say p-adic numbers such that each coordinate axes in the extension corresponds to single infinite integer of form $N = mM$. Thus the most general infinite integer N would have the form

$$N = m_0 + \sum m_i M_i \ . \tag{15}$$

This representation of infinite integers indeed looks promising from the point of view of practical calculations. The representation looks also attractive physically. One can interpret the set of integers N as a linear space with integer coefficients m_0 and m_i:

$$N = m_0|1\rangle + \sum m_i|M_i\rangle \ . \tag{16}$$

$|M_i\rangle$ can be interpreted as a state basis representing many-particle states formed from bosons labelled by infinite primes p_k and $|1\rangle$ represents Fock vacuum. Therefore this representation is analogous to a quantum superposition of bosonic Fock states with integer, rather than complex valued, superposition coefficients. If one interprets M_i as orthogonal state basis and interprets m_i as p-adic integers, one can define inner product as

$$\langle N_a, N_b\rangle = m_0(a)m_0(b) + \sum_i m_i(a)m_i(b) \ . \tag{17}$$

This expression is well defined p-adic number if the sum contains only enumerable number of terms and is always bounded by p-adic ultrametricity. It converges if the p-adic norm of of m_i approaches to zero when M_i increases.

6.8.2 Generalized rationals

Generalized rationals could be defined as ratios $R = M/N$ of the generalized integers. This works nicely when M and N are expressible as products of powers of finite or infinite primes but for more general integers the definition does not look attractive. This suggests that one should restrict the generalized rationals to be numbers having the expansion as a product of positive and negative primes, finite or infinite:

$$N = \prod_k p_k^{n_k} = \frac{n_1 M_1}{nM} \quad . \tag{18}$$

6.8.3 Generalized reals form infinite-dimensional real vector space

One could consider the possibility of defining generalized reals as limiting values of the generalized rationals. A more practical definition of the generalized reals is based on the generalization of the pinary expansion of ordinary real number given by

$$\begin{aligned} x &= \sum_{n \geq n_0} x_n p^{-n} \ , \\ x_n &\in \{0, .., p-1\} \ . \end{aligned} \tag{19}$$

It is natural to try to generalize this expansion somehow. The natural requirement is that sums and products of the generalized reals and canonical identification map from the generalized reals to generalized p-adcs are readily calculable. Only in this manner the representation can have practical value.

These requirements suggest the following generalization

$$\begin{aligned} X &= x_0 + \sum_N x_N p^{-N} \ , \\ N &= \sum_i m_i M_i \ , \end{aligned} \tag{20}$$

where x_0 and x_N are ordinary reals. Note that N runs over infinite integers which has *vanishing finite part*. Note that generalized reals can be regarded as infinite-dimensional linear space such that each infinite integer N corresponds to one coordinate axis of this space. One could interpret generalized real as a superposition of bosonic Fock states formed from single single boson state labelled by prime p such that occupation number is either 0 or infinite integer N with a vanishing finite part:

$$X = x_0|0\rangle + \sum_N x_N |N> \quad . \tag{21}$$

The natural inner product is

$$\langle X,Y\rangle \quad = \quad x_0y_0+\sum_N x_Ny_N \quad . \tag{22}$$

The inner product is well defined if the number of N:s in the sum is enumerable and x_N approaches zero sufficiently rapidly when N increases. Perhaps the most natural interpretation of the inner product is as R_p valued inner product.

The sum of two generalized reals can be readily calculated by using only sum for reals:

$$X+Y \quad = \quad x_0+y_0+\sum_N (x_N+y_N)p^{-N} \quad , \tag{23}$$

The product XY is expressible in the form

$$XY \quad = \quad x_0y_0+x_0Y+Xy_0+\sum_{N_1,N_2} x_{N_1}y_{N_2}p^{-N_1-N_2} \quad , \tag{24}$$

If one assumes that infinite integers form infinite-dimensional vector space in the manner proposed, there are no problems and one can calculate the sums N_1+N_2 by summing component wise manner the coefficients appearing in the sums defining N_1 and N_2 in terms of infinite integers M_i allowing expression as a product of infinite integers.

Canonical identification map from ordinary reals to p-adics

$$x=\sum_k x_kp^{-k} \quad \rightarrow \quad x_p=\sum_k x_kp^k \quad ,$$

generalizes to the form

$$x \quad = \quad x_0+\sum_N x_Np^{-N} \quad \rightarrow \quad (x_0)_p+\sum_N (x_N)_pp^N \quad , \tag{25}$$

so that all the basic requirements making the concept of generalized real calculationally useful are satisfied.

There are several interesting questions related to generalized reals.

a) Are the extensions of reals defined by various values of p-adic primes mathematically equivalent or not? One can map generalized reals associated with various choices of the base p to each other in one-one manner using the mapping

$$X = x_0 + \sum_N x_N p_1^{-N} \rightarrow x_0 + \sum_N x_N p_2^{-N} . \tag{26}$$

The ordinary real norms of *finite* (this is important!) generalized reals are identical since the representations associated with different values of base p differ from each other only infinitesimally. This would suggest that the extensions are physically equivalent. It these extensions are not mathematically equivalent then p-adic primes could have a deep role in the definition of the generalized reals.

b) One can generalize previous formulas for the generalized reals by replacing the coefficients x_0 and x_i by complex numbers, quaternions or octonions so as to get generalized complex numbers, quaternions and octonions. Also inner product generalizes in an obvious manner. The 8-dimensionality of the imbedding space provokes the question whether it might be possible to regard the infinite-dimensional configuration space of 3-surfaces, or rather, its tangent space, as a Hilbert space realization of the generalized octonions. This kind of identification could perhaps reduce TGD based physics to generalized number theory.

6.9 Comparison with the approach of Cantor

The main difference between the approach of Cantor and the proposed approach is that Cantor uses only the basic arithmetic concepts such as sum and multiplication and the concept of successor defining ordering of both finite and infinite ordinals. Cantor's approach is also purely set theoretic. The problems of purely set theoretic approach are related to the question what the statement 'Set is Many allowing to regard itself as One' really means and to the fact that there is no obvious connection with physics. The proposed approach is based on the introduction of the concept of prime as a basic concept whereas ordering is based on the use of ratios: using these one can recursively define ordering and get precise quantitative information based on finite reals. Together with canonical identification the concept of infinite primes becomes completely physical in the sense that all probabilities are always finite real numbers. The 'Set is Many allowing to regard itself as One' is defined as quantum physicist would define it: many particle states become single particle states in the second quantization describing the counterpart for the construction of the set of subsets of a given set. One could also say that integer as such corresponds to set as 'One' and its decomposition to a product of primes corresponds to the set as 'Many'. The concept of prime, the ultimate 'One', has as its physical counterpart the concept of elementary particle understood in very general sense. The new element is the physical interpretation: the sum of two numbers whose ratio is zero correspond to completely physical finite-subsystem-infinite complement division and the iterated construction of the set of subsets of a set at given level is basically p-adic evolution understood in the most general possible sense and realized as a repeated second quantization. What is attractive is that this repeated second quantization can be regarded also as a model of abstraction process and actually the process of abstraction itself.

The possibility to interpret the construction of infinite primes either as a repeated bosonic quantization involving subsystem-complement division or as a repeated super-symmetric quantization could have some deep meaning. A possible interpretation consistent with these two pictures is based on the hypothesis that fermions provide a reflective level of consciousness in the sense that the 2^N element Fock basis of many-fermion states formed from N single-fermion states can be regarded as a set of all possible statements about N basic statements. Statements about whether a given element of set X belongs to some subset S of X are certainly the fundamental statements from the point of view of mathematics. Hence one could argue that many-fermion states provide cognitive representation for the subsets of some set. Single fermion states represent the points of the set and many-fermion states represent possible subsets.

References

[TGDview] M. Pitkänen (2006), *Topological Geometrodynamics: Overview.* http://www.physics.helsinki.fi/∼matpitka/tgdview/tgdview.html.

[TGDgeom] M. Pitkänen (2006), *Quantum Physics as Infinite-Dimensional Geometry.* http://www.physics.helsinki.fi/∼matpitka/tgdgeom/tgdgeom.html.

[TGDquant] M. Pitkänen (2006), *Quantum TGD.* http://www.physics.helsinki.fi/∼matpitka/tgdquant/tgdquant.html.

[TGDclass] M. Pitkänen (2006), *Physics in Many-Sheeted Space-Time.* http://www.physics.helsinki.fi/∼matpitka/tgdclass/tgdclass.html.

[TGDnumber] M. Pitkänen (2006), *TGD as a Generalized Number Theory.* http://www.physics.helsinki.fi/∼matpitka/tgdnumber/tgdnumber.html.

[TGDpad] M. Pitkänen (2006), *p-Adic length Scale Hypothesis and Dark Matter Hierarchy.* http://www.physics.helsinki.fi/∼matpitka/paddark/paddark.html.

[TGDfree] M. Pitkänen (2006), *TGD and Fringe Physics.* http://www.physics.helsinki.fi/∼matpitka/freenergy/freenergy.html.

[TGDconsc] M. Pitkänen (2006), *TGD Inspired Theory of Consciousness.* http://www.physics.helsinki.fi/∼matpitka/tgdconsc/tgdconsc.html.

[TGDselforg] M. Pitkänen (2006), *Bio-Systems as Self-Organizing Quantum Systems.* http://www.physics.helsinki.fi/∼matpitka/bioselforg/bioselforg.html.

[TGDware] M. Pitkänen (2006), *Quantum Hardware of Living Matter.* http://www.physics.helsinki.fi/∼matpitka/bioware/bioware.html.

[TGDholo] M. Pitkänen (2006), *Bio-Systems as Conscious Holograms.* http://www.physics.helsinki.fi/∼matpitka/hologram/hologram.html.

[TGDgeme] M. Pitkänen (2006), *Mathematical Aspects of Consciousness Theory.* http://www.physics.helsinki.fi/∼matpitka/genememe/genememe.html.

[TGDeeg] M. Pitkänen (2006), *TGD and EEG.* http://www.physics.helsinki.fi/∼matpitka/tgdeeg/tgdeeg/tgdeeg.html.

[TGDmagn] M. Pitkänen (2006), *Magnetospheric Consciousness.* http://www.physics.helsinki.fi/∼matpitka/magnconsc/magnconsc.html.

[TGDmath] M. Pitkänen (2006), *Mathematical Aspects of Consciousness Theory.* http://www.physics.helsinki.fi/∼matpitka/mathconsc/mathconsc.html.

[1] J. Esmonde and M. Ram Murty (1991), *Problems in Algebraic Number Theory*, Springer-Verlag, New York.

[2] T. Smith (1997), *D4-D5-E6 Physics*. Homepage of Tony Smith. *http : //galaxy.cau.edu/tsmith/d4d5e6hist.html*. The homepage contains a lot of information and ideas about the possible relationship of octonions and quaternions to physics.

[3] J. Daboul and R. Delborough (1999) *Matrix Representations of Octonions and Generalizations*, hep-th/9906065.

[4] J. Schray and C. A. Manogue (1994) *Octonionic representations of Clifford algebras and triality*, hep-th/9407179.

[5] R. Harvey (1990), *Spinors and Calibrations*, Academic Press, New York.

[6] S. S. Abhyankar (1980), *Algebraic Geometry for Scientists and Engineers*, Mathematical Surveys and Monographs, No 35, American Mathematical Society.

[7] A. Robinson (1974), *Nonstandard Analysis*, North-Holland, Amsterdam.

[B4] The chapter *Configuration Space Spinor Structure* of [TGDgeom]. http://www.physics.helsinki.fi/∼matpitka/tgdgeom/tgdgeom.html#cspin.

[C1] The chapter *Construction of Quantum Theory* of [TGDquant]. http://www.physics.helsinki.fi/∼matpitka/tgdquant/tgdquant.html#quthe.

[C5] The chapter *Equivalence of Loop Diagrams with Tree Diagrams and Cancellation of Infinities in Quantum TGD* of [TGDquant]. http://www.physics.helsinki.fi/∼matpitka/tgdquant/tgdquant.html#bialgebra.

[C6] The chapter *Was von Neumann Right After All* of [TGDquant]. http://www.physics.helsinki.fi/∼matpitka/tgdquant/tgdquant.html#vNeumann.

[D1] The chapter *Basic Extremals of Kähler Action* of [TGDclass]. http://www.physics.helsinki.fi/∼matpitka/tgdclass/tgdclass.html#class.

[D5] The chapter *TGD and Cosmology* of [TGDclass]. http://www.physics.helsinki.fi/∼matpitka/tgdclass/tgdclass.html#cosmo.

[E10] The chapter *Intentionality, Cognition, and Physics as Number theory or Space-Time Point as Platonia* of [TGDnumber]. http://www.physics.helsinki.fi/∼matpitka/tgdnumber/tgdnumber.html#intcognc.

[E1] The chapter *TGD as a Generalized Number Theory: p-Adicization Program* of [TGDnumber].
http://www.physics.helsinki.fi/∼matpitka/tgdnumber/tgdnumber.html#visiona.

[E2] The chapter *TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts* of [TGDnumber].
http://www.physics.helsinki.fi/∼matpitka/tgdnumber/tgdnumber.html#visionb.

[E3] The chapter *TGD as a Generalized Number Theory: Infinite Primes* of [TGDnumber].
http://www.physics.helsinki.fi/∼matpitka/tgdnumber/tgdnumber.html#visionc.

[E4] The chapter *p-Adic Numbers and Generalization of Number Concept* of [TGDnumber].
http://www.physics.helsinki.fi/∼matpitka/tgdnumber/tgdnumber.html#padmat.

[E8] The chapter *Riemann Hypothesis and Physics* of [TGDnumber].
http://www.physics.helsinki.fi/∼matpitka/tgdnumber/tgdnumber.html#riema.