Equivalence of Loop Diagrams with Tree Diagrams and Cancellation of Infinities in Quantum TGD

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Abstract

The great dream of a physicist believing in reductionism and TGD would be a formalism generalizing Feynman diagrams allowing any graduate student to compute the predictions of the theory. TGD has forced myself to give up naive reductionism but I believe that TGD allows generalization of Feynman diagram in such a manner that one gets rid of the infinities plaguing practically all existing theories. The purpose of this chapter is to develop general vision about how this might be achieved. The vision is based on generalization of mathematical structures discovered in the construction of topological quantum field theories (TQFT), conformal field theories (CQFT). In particular, the notions of Hopf algebras and quantum groups, and categories are central. The following gives a very concise summary of the basic ideas.

1. Feynman diagrams as generalized braid diagrams

The first key idea is that generalized Feynman diagrams with diagrams which are analogous to knot and link diagrams in the sense that diagrams involving loops are equivalent with tree diagrams. This would be a generalization of duality symmetry of string models.

TGD itself provides general arguments supporting same idea. The identification of absolute minimum of Kähler action as a four-dimensional Feynman diagram characterizing particle reaction means that there is only single Feynman diagram instead of functional integral over 4-surfaces: this diagram is expected to be minimal one. S-matrix element as a representation of a path defining continuation of configuration space spinor field between different sectors of it corresponding different 3-topologies leads also to the conclusion that all continuations and corresponding Feynman diagrams are equivalent. Universe as a compute metaphor idea allowing quite concrete realization by generalization of what is meant by space-time point leads to the view that generalized Feynman diagrams characterize equivalent computations.

2. Coupling constant evolution from infinite number of critical values of Kähler coupling strength

The basic objection that this vision does not allow to understand coupling constant evolution involving loops in an essential manner can be circumvented. Quantum criticality requires that Kähler coupling constant α_K is analogous to critical temperature (so that the loops for configuration space integration vanish). The hypothesis motivated by the enormous vacuum degeneracy of Kähler action is that α_K has an infinite number of possible values labelled by p-adic length scales and also probably also by the dimensions of effective tensor factors defined hierarchy of II_1 factors (so called Beraha numbers) as found already earlier. The dependence on p-adic length scale L_p corresponds to the usual renormalization group evolution whereas the latter dependence would correspond to angular resolution and finite-dimensional extensions of p-adic number fields R_p. Finite finite resolution and renor-

malization group evolution are forced by the algebraic continuation of rational number based physics to real and p-adic number fields since p-adic and real notions of distance between rational points differ dramatically.

3. R-matrices, complex numbers, quaternions, and octonions

A crucial observation is that physically equivalent R-matrices of 6-vertex models are labelled by points of CP_2 whereas maximal space of commuting R-matrices are labelled by points of 2-sphere S^2. CP_2 also labels maximal associative and thus quaternionic subspaces of 8-dimensional octonion space whereas S^2 labels the maximally commutative subspaces of quaternion space, which suggests that R-matrices and these structures correspond to each other.

Number theoretic vision leads to a number theoretic variant of spontaneous compactification meaning that space-time surfaces could be regarded either as hyper-quaternionic, and thus maximal associative, 4-surfaces in M^8 regarded as the space of hyper-octonions or as surfaces in $M^4 \times CP_2$ (the imaginary units of hyper-octonions are multiplied with $\sqrt{-1}$ so that the number theoretical norm has Minkowski signature). Associativity constraint is an essential element of also Yang-Baxter equations.

These observations lead to a concrete proposal how how quantum classical correspondence is realized classically at space-time level. Each point of CP_2 corresponds to R-matrix and 3-surfaces can be identified as preferred sections of space-time surface by requiring that unitary R-matrices are commuting in the section (micro-causality) whereas commutativity fails for R-matrices corresponding to different values of time coordinate defined by this foliation.

4. Ordinary conformal symmetries act on the space of super-canonical conformal weights

TGD predicts two kinds of super-conformal symmetries.

a) Quaternion conformal symmetries define super Kac-Moody representations and are realized at light-like 3-surfaces appearing as boundaries of space-time sheets and boundaries between space-time regions with Euclidian and Minkowskian signature of metric. Conformal weights are half integer valued in this case.

b) Super-canonical conformal invariance acts at the level of imbedding space and corresponds to light-like 7-surfaces of form $X^3_l \times CP_2 \subset M^4 \times CP_2$ believed to appear as causal determinants too. In this case conformal weights are complex numbers of form $\Delta = n/2 + iy$ and for physical states they are expected to reduce to the form $\Delta = 1/2 + iy$.

Quantum classical correspondence suggests that the complex conformal weights of super-canonical algebra generators have space-time counterparts. The proposal is that the weights are mapped to the points of geodesic sphere of CP_2 (and thus also of space-time surface) labelling also mutually commutating R-matrices. The map is completely analogous to the map of momenta of quantum particles to the points of celestial sphere. One can thus regard super-generators as conformal fields in space-time or complex plane in which conformal weights appear as punctures. The action of super-conformal algebra and braid group on these points realizing monodromies of conformal field theories induces by a pull-back a braid group action on the conformal labels of configuration space gamma matrices (super generators) and corresponding isometry generators.

The gamma matrices and isometry generators spanning the super-canonical algebra can be regarded as fields of a conformal field theory in the complex plane containing infinite number of punctures defined by the complex conformal weights. Quaternion conformal Kac Moody and Super Virasoro algebras act as symmetries of this theory and S-matrix of TGD should involve the n-point functions of this conformal field theory.

This picture also justifies the earlier proposal that configuration space Clifford algebra defined by the gamma matrices acting as super generators defines an infinite-dimensional von Neumann algebra possessing hierarchies of type II_1 factors having a close connection with non-trivial representations of braid group and quantum groups. The zeros of Riemann Zeta along the line $Re(s) = 1/2$ in the plane of conformal weights could be regarded a the infinite braid behind the von Neumann algebra. Contrary to the expectations, also trivial zeros seem to be important. Super-canonical algebra predicts that also the superposition for the imaginary parts of non-trivial zeros makes sense so that one would have an entire hierarchy of one-dimensional lattices depending on the number of non-trivial zeros included. The braids defined by sub-lattices defined by subsets of non-trivial zeros could be seen as completely integrable 1-dimensional spin chains leading to a hierarchy of quantum groups and braiding matrices naturally.

It seems that not only Riemann's zeta but also polyzetas could play a fundamental role in TGD Universe. The super-canonical conformal weights of interacting particles, in particular of those forming bound states, are expected to have "off mass shell" values. An attractive hypothesis is that they correspond to the zeros of Riemann's polyzetas. Interaction would allow quite concretely the realization of braiding operations dynamically.

5. Equivalence of loop diagrams with tree diagrams from the axioms of generalized ribbon category

The fourth idea is that Hopf algebra related structures and appropriately generalized ribbon categories could provide a concrete realization of this picture. Generalized Feynman diagrams which are identified as braid diagrams with strands running in both directions of time and containing besides braid operations also boxes representing algebra morphisms with more than one incoming and outgoing strands describing particle reactions (3-particle vertex should be enough). In particular, fusion of 2-particles and decay of particle to two would correspond to generalizations of algebra product μ and co-product Δ to morphisms of the category defined by the super-canonical algebras associated with 3-surfaces with various topologies and conformal structures. The basic axioms for this structure generalizing Hopf algebra axioms state that diagrams with self energy loops, vertex corrections, and box diagrams are equivalent with tree diagrams.

6. Quantum criticality and renormalization group invariance

Quantum criticality means that renormalization group acts like isometry group at a fixed point rather than acting like a gauge symmetry as in the standard quantum field theory context. Despite this difference it is possible to understand how Feynman graph expansion with vanishing loop corrections relates to generalized Feynman graphs and a nice connection with the Hopf- and Lie algebra structures assigned by Connes and Kreimer to Feynman graphs emerges. The condition that loop diagrams are equivalent with tree diagrams gives explicit equations which might fix completely also the p-adic length scale evolution of vertices. Quantum criticality in principle fixes completely the values of the masses and coupling constants as a function of p-adic length scale.

7. The spectrum of zeros of Zeta and quantum TGD

The question whether the imaginary parts of the Riemann Zeta are linearly independent (as assumed in the previous work) or not is of crucial physical significance. Linear independence implies that the spectrum of the super-canonical weights is essentially an infinite-dimensional lattice. Otherwise a more complex structure results.

The hypothesis that p^{iy} is an algebraic phase for every prime implies that p^{iy} is expressible as a product of a Pythagorean phase and a root of unity for every prime p. The numerical evidence supporting the translational invariance of the correlations for the spectrum of zeros together with p-adic considerations leads to the working hypothesis that for any prime p one can express the spectrum of zeros as the product of a subset of Pythagorean prime phases and of a fixed subset U of roots of unity. The spectrum of zeros could be expressed as a union over the translates of the same basic spectrum defined by the roots of unity: this is consistent with what the spectral correlations strongly suggest. That decompositions defined by different primes p yield the same spectrum would mean a powerful number theoretical symmetry realizing p-adicities at the level of the spectrum of Zeta.

The model for the scalar propagator as a partition function for the super-canonical algebra supports this view. The approximation that the zeros are linearly independent implies for any subalgebra defined by a finite set of zeros a universal spectrum of singularities for real values of mass squared, and at the limit of entire algebra the propagator is not mathematically defined since the singularities are dense in real axis. Linear independence however shifts the poles to complex values of mass and genuine resonances result and the propagator is expected to be well-defined for the entire algebra. The theory predicts universal momentum and p-adic length scale dependence of propagators in this picture and the option based on super-canonical algebra predicts very simple and elegant expression for the propagator with all the desired properties.

8. Quantum field theory formulation of quantum TGD

The super-canonical generalization of a 2-dimensional conformal field theory seems to be indispensable for the construction of S-matrix at the fundamental level and defines the vertices as n-point functions of a conformal field theory in turn used to construct S-matrix as tree diagrams. Also a quantum field theory defined for 3-surface X^3 belonging to the light like 7-surfaces $X^3_l \times CP_2$ defining causal determinants seems to make sense. The QFT in question is determined by the absolute minimum $X^4(X^3)$ associated with the maximum of Kähler function and is consistent with the loop corrections in configuration space. The restriction to X^3 realizes quantum holography and means that a minimum amount of data about $X^4(X^3)$ is needed.

The modified Dirac action for the induced spinor fields at the maximum of the Kähler function with induced spinor fields identified as Grassmann algebra valued fields provides the sought for action. The fermionic propagator is defined by the inverse of the modified Dirac operator. Bosonic kinetic term is Grassmann algebra valued and vanishes for $\psi = 0$ and contributes nothing to the perturbation series in accordance with effective freezing of the configuration space degrees of freedom. Since the fermionic action is free action, a divergence free quantum field theory is in question. One could also see the action as a fixed point of the map sending action to effective action in accordance with the idea that loop corrections vanish. Different sub-algebras of the super-canonical algebra correspond naturally to various conformally inequivalent configuration space metrics and infinite-dimensional sub-algebras correspond to the singular limits when the space-time surface becomes vacuum extremal so that the modified Dirac operator vanishes identically and super-canonical propagator becomes ill-defined.

1 Introduction

The great dream of a physicist believing in reductionism and TGD would be a formalism generalizing Feynman diagrams allowing any graduate student to compute the predictions of the theory. TGD has forced myself to give up naive reductionism but I believe that TGD allows a generalization of

Feynman diagrammatics free of the infinities plaguing practically all existing theories. The purpose of this chapter is to develop a general vision about how this might be achieved. The vision is based on generalization of mathematical structures discovered in the construction of topological quantum field theories (TQFT) and conformal field theories (CQFT). In particular, the notions of Hopf algebras and quantum groups, and categories are central. The following gives a very concise summary of the basic ideas.

1.1 Feynman diagrams as generalized braid diagrams

The first key idea is that generalized Feynman diagrams are analogous to knot and link diagrams in the sense that they allow also "moves" allowing to identify classes of diagrams and that the diagrams containing loops are equivalent with tree diagrams, so that there would be no summation over diagrams. This would be a generalization of duality symmetry of string models.

TGD itself provides general arguments supporting same idea. The identification of absolute minimum of Kähler action as a four-dimensional Feynman diagram characterizing particle reaction means that there is only single Feynman diagram instead of functional integral over 4-surfaces: this diagram is expected to be minimal one. At quantum level S-matrix element can be seen as a representation of a path defining continuation of configuration space (CH) spinor field between different sectors of CH corresponding to different 3-topologies. All continuations and corresponding Feynman diagrams are equivalent. The idea about Universe as a computer and algebraic hologram allows a concrete realization based on the notion of infinite primes, and space-time points become infinitely structured monads [E10]. The generalized Feynman diagrams differing only by loops are equivalent since they characterize equivalent computations.

1.2 Coupling constant evolution from quantum criticality

The basic objection against the new view about Feynman diagrams is that it is not consistent with the notion of coupling constant evolution involving loops in an essential manner. The objection can be circumvented. Quantum criticality requires that Kähler coupling constant α_K is analogous to critical temperature (so that the loops for configuration space integration vanish) and thus highly unique.

1. A hypothesis motivated by the enormous vacuum degeneracy of Kähler action was that α_K has an infinite number of possible values labelled

by p-adic length scales and possibly also by the dimensions of effective tensor factors defined hierarchy of II_1 factors (so called Beraha numbers) as found in [E10, C7]. The dependence on p-adic length scale L_p would correspond to the usual renormalization group evolution whereas the latter dependence would correspond to a finite angular resolution and to a hierarchy of finite-dimensional extensions of p-adic number fields R_p. The finiteness of the resolution is forced by the algebraic continuation of rational number based physics to real and p-adic number fields since p-adic and real notions of distance between rational points differ dramatically. The higher the algebraic dimension of the extension and the higher the value of p-adic prime the better the angular (or phase) resolution and nearer the p-adic topology to that for real numbers.

p-Adic renormalization group invariance of the gravitational constant led to an explicit model for the dependence of α_K on p. This option has not led to an understanding of electro-weak and color coupling evolutions: the basic problem is that the increase of α_K at high energies is predicted to be too fast.

2. The most stringent hypothesis is that only single value of α_K is possible and this was indeed the original hypothesis. The reason for giving it up was that the p-adic evolution of the gravitational coupling strength was predicted to evolve as $G \propto L_p^2$ and thus quite too rapidly. However, if gravitons correspond to the largest non-super-astrophysical Mersenne prime M_{127}, gravitational interactions are mediated by space-time sheets with effective p-adic topology characterized by M_{127} and gravitational coupling is effectively RG invariant. One can also correlate the p-adic evolution of color and electro-weak couplings [C4] in a successful manner and to identify the value of α_K as the value of electro-weak $U(1)$ coupling for Mersenne prime M_{127}. Therefore it seems to safe to conclude that the situation is completely settled now.

1.3 R-matrices, complex numbers, quaternions, and octonions

A crucial observation is that physically equivalent R-matrices of 6-vertex models are labelled by points of CP_2 whereas maximal space of commutating R-matrices are labelled by the points of 2-sphere S^2 [19]. As found in [E10, E2], CP_2 also labels maximal associative and thus quaternionic sub-spaces of

8-dimensional octonion space. S^2 in turn labels the maximally commutative sub-spaces of quaternion space, which suggests that R-matrices and these structures correspond to each other.

Number theoretic vision leads to a number theoretic variant of spontaneous compactification meaning that space-time surfaces could be regarded either as hyper-quaternionic, and thus maximal associative, 4-surfaces in M^8 regarded as the space of hyper-octonions or as surfaces in $M^4 \times CP_2$ (the imaginary units of hyper-octonions are multiplied with $\sqrt{-1}$ so that the number theoretical norm has Minkowski signature)[E2]. Associativity constraint is an essential element of also Yang-Baxter equations [18, 19].

These observations lead to a concrete proposal how quantum classical correspondence is realized classically at the space-time level. Each point of CP_2 corresponds to R-matrix and 3-surfaces can be identified as preferred sections of the space-time surface by requiring that unitary R-matrices are commuting in the section (micro-causality) whereas commutativity fails for R-matrices corresponding to different values of time coordinate defined by this foliation.

1.4 Ordinary conformal symmetries act on the space of super-canonical conformal weights

TGD predicts two kinds of super-conformal symmetries.

1. The ordinary super-conformal symmetries realized at the space-time level are associated with super Kac-Moody representations realized at light-like 3-surfaces appearing as boundaries of space-time sheets and boundaries between space-time regions with Euclidian and Minkowskian signature of metric. Conformal weights are half-integer valued in this case.

2. Super-canonical conformal invariance acts at the level of imbedding space and corresponds to light-like 7-surfaces of form $X_l^3 \times CP_2 \subset M^4 \times CP_2$ believed to appear as causal determinants too. In this case conformal weights are complex numbers of form $\Delta = n/2 + iy$ for the generators if super algebra, and I have proposed that the conformal weights of the physical states correspond are of form $1/2 + iy$, where y is zero of Riemann Zeta or possibly even superposition of them. In the latter case the imaginary parts of zeros would define a basis of an infinite-dimensional Abelian group spanning the weights. Later it will be found that quantum criticality favors zeros but not superpositions of their imaginary parts.

The basic question concerns the interaction of these conformal symmetries.

1. Quantum classical correspondence suggests that the complex conformal weights of super-canonical algebra generators have space-time counterparts. The proposal is that the weights are mapped to the points of geodesic sphere of CP_2 (and thus also of space-time surface) labelling also mutually commuting R-matrices. The map is completely analogous to the map of momenta of quantum particles to the points of celestial sphere. One can thus regard super-generators as conformal fields in space-time or complex plane having conformal weights as punctures. The action of super-conformal algebra and braid group on these points realizing monodromies of conformal field theories [19] induces by a pull-back a braid group action on the conformal labels of configuration space gamma matrices (super generators) and corresponding isometry generators.

2. The gamma matrices and isometry generators spanning the super-canonical algebra can be regarded as fields of a conformal field theory in the complex plane containing infinite number of punctures defined by the complex conformal weights. Quaternion conformal Super Virasoro algebra and Kac Moody algebra would act as symmetries of this theory and the S-matrix of TGD would involve the n-point functions of this conformal field theory.

This picture also justifies the earlier proposal that configuration space Clifford algebra defined by the gamma matrices acting as super generators defines an infinite-dimensional von Neumann algebra possessing hierarchies of type II_1 factors [20] having a close connection with the non-trivial representations of braid group and quantum groups. The sequence of non-trivial zeros of Riemann Zeta along the line $Re(s) = 1/2$ in the plane of conformal weights could be regarded an an infinite braid behind the von Neumann algebra [20]. Contrary to the expectations, also trivial zeros seem to be important. Purely algebraic considerations support the view that superposition for the imaginary parts of non-trivial zeros makes sense so that one would have entire hierarchy of one-dimensional lattices depending on the number of zeros included. The finite braids defined by subsets of zeros could be seen as a hierarchy of completely integrable 1-dimensional spin chains leading to quantum groups and braid groups [18, 19] naturally. In conformal field theories it is possible to construct explicitly the generators of quantum group in terms of operators creating screening charges [19]: interestingly, the charges are located going along a line parallel to imaginary axis to infinity.

It seems that not only Riemann's zeta but also polyzetas [21, 22, 28, 29] could play a fundamental role in TGD Universe. The super-canonical conformal weights of interacting particles, in particular of those forming bound states, are expected to have "off mass shell" values. An attractive hypothesis is that they correspond to zeros of Riemann's polyzetas. Interaction would allow quite concretely the realization of braiding operations dynamically. The physical justification for the hypothesis would be quantum criticality. Indeed, it has been found that the loop corrections of quantum field theory are expressible in terms of polyzetas [23]. If the arguments of polyzetas correspond to conformal weights of particles of many-particle bound state, loop corrections vanish when the super-canonical conformal weights correspond to the zeros of polyzetas including zeta. This argument does not allow superpositions of imaginary parts of zeros.

1.5 Equivalence of loop diagrams with tree diagrams from the axioms of generalized ribbon category

The fourth idea is that Hopf algebra related structures and appropriately generalized ribbon categories [18, 19] could provide a concrete realization of this picture. Generalized Feynman diagrams which are identified as braid diagrams with strands running in both directions of time and containing besides braid operations also boxes representing algebra morphisms with more than one incoming and outgoing strands. 3-particle vertex should be enough, and the fusion of 2-particles and $1 \rightarrow 2$ particle decay would correspond to generalizations of the algebra product μ and co-product Δ to morphisms of the category defined by the super-canonical algebras associated with 3-surfaces with various topologies and conformal structures. The basic axioms for this structure generalizing ribbon algebra axioms [18] would state that diagrams with self energy loops, vertex corrections, and box diagrams are equivalent with tree diagrams.

Tensor categories might provide a deeper understanding of p-adic length scale hypothesis. Tensor primes can be identified as vector/Hilbert spaces, whose real or complex dimension is prime. They serve as "elementary particles" of tensor category since they do not allow a decomposition to a tensor product of lower-dimensional vector spaces. The unit I of the tensor category would have an interpretation as a one-dimensional Hilbert space or as the number field associated with the Hilbert space and would act like identity with respect to tensor product. Quantum jump cannot decompose tensor prime system to an unentangled product of sub-systems. This elementary particle like aspect of tensor primes might directly relate to the origin of p-adicity. Also infinite primes are possible and could distinguish between different infinite-dimensional state spaces.

For quantum dimensions $[n]_q \equiv (q^n - q^{-n})/(q - q^{-1})$ [18] no decomposition into a product of prime quantum dimensions exist and one can say that all non-vanishing quantum integers $[n]_q$ are primes. For q an n^{th} root of unity, quantum integers form a finite set containing only the elements $0, 1, ..., [n-1]$ so that quantum dimension is always finite. The numbers $[2]_q^2$, for $q = exp(i\pi/n)$ define a hierarchy of Beraha numbers having an interpretation as a renormalized dimension $[2]_q^2 \leq 4$ for the Clifford algebra of 2-dimensional space and appearing as effective dimensions of type II_1 sub-factors of von Neumann algebras.

1.6 What about loop diagrams with a non-singular homologically non-trivial imbedding to a Riemann surface of minimal genus?

There is an objection against the reduction to tree diagrams. Photon-photon scattering rate vanishes classically (zeroth order in $\hbar$), and the lowest non-vanishing contribution to the scattering amplitude corresponds to a box diagram and diagrams obtained from it by permuting the outgoing photon lines. All diagrams that can have intersecting lines as planar diagrams when outgoing lines are permuted, can be imbedded to a Riemann surface with some minimal genus as diagrams without intersecting lines containing a minimal number of homologically non-trivial loops.

This suggests a more general axiom: diagrams are characterized by genus and only homologically trivial loops can be eliminated. Also the question arises whether a sum over the different genera must be assumed as string model would suggest or whether single value of genus is enough. The discussion of this chapter assumes a reduction to a tree diagram but the generalization allowing all values of genus should be rather straightforward.

1.7 Quantum criticality and renormalization group invariance

Quantum criticality means that renormalization group acts like isometry group at a fixed point rather than acting like a gauge symmetry as in the standard quantum field theory context. Despite this difference it is possible to understand how Feynman graph expansion with vanishing loop corrections relates to generalized Feynman graphs and a nice connection with the Hopf- and Lie algebra structures assigned by Connes and Kreimer to Feynman graphs emerges. It is possible to deduce an explicit representation for the universal momentum and p-adic length scale dependence of propagators in this picture. The condition that loop diagrams are equivalent with tree diagrams gives explicit equations which might fix completely also the p-adic length scale evolution of vertices. Quantum criticality in principle fixes completely the values of the masses and coupling constants as a function of p-adic length scale.

To sum up, although these new ideas are rather speculative and my poor algebraic skills restrict severely the attempts to develop them into a more detailed theory, I believe that the new vision makes a big step in concretizing the physics as a generalized number theory vision. There is a lot of work to do. The implications for the construction of configuration space geometry and spinor structure are expected to be highly non-trivial and should tighten the loose web of ideas about the relationship and role of super-canonical and Super-Kac-Moody algebras. The new ideas relate also closely to the p-adicization program being partly inspired by it. Also the fascinating idea that $M^4 \times CP_2$ a emerges naturally from the interpretation of space-time surfaces as maximally associative 4-manifolds of octonionic space should be developed further.

2 Generalizing the notion of Feynman diagram

In this section various motivations for generalizing the notion of Feynman diagram such that diagrams with loops are equivalent with tree diagrams are discussed.

2.1 Divergence cancellation mechanisms in TGD

TGD provides general mechanisms for the cancellation of ultraviolet divergences.

1. The standard divergence due to the Gaussian determinant is absent in TGD. The Kähler geometry of the configuration space implies a cancellation of the configuration space metric determinant and Gaussian determinant associated with the integral over the quantum fluctuating configuration space degrees of freedom by Gaussian integration around a maximum of Kähler function.

2. The divergences due to the 3-dimensional micro-locality are absent due to the generalization of point particle which means that state functionals are non-local functionals of the 3-surface and local interaction vertices are smoothed out. Locality is realized at the configuration space level but there is now second quantization at this level since physical states correspond to classical configuration space spinor fields.

3. The vacuum energy for induced second quantized spinor fields cancels when fermions and anti-fermions have opposite sign of inertial energy. Crossing symmetry allows vanishing of the net inertial energy at least in cosmological length scales and even shorter length scales assuming that gravitational four-momentum is the difference of 4-momenta associated with matter and antimatter with an appropriate sign factor taking care that gravitational energy is positive.

4. The infinite-dimensional space of zero modes does not allow any natural integration measure. If localization in zero modes occurs in each quantum jump and leads to a discrete subspace of zero modes, the integration over zero modes becomes trivial. The localization can be interpreted in terms of a quantum measurement correlating classical macroscopic degrees of freedom represented by zero modes correlated with quantum fluctuating degrees of freedom. Entanglement need not be reduced completely since discrete subset of zero modes can remain entangled in this manner.

2.2 Motivation for generalized Feynman diagrams from topological quantum field theories and generalization of string model duality

There are many manners to end up to the idea that generalized Feynman diagrams are analogous to knot and link diagrams and that diagrams with loops are equivalent with tree diagrams.

2.2.1 Topological quantum field theories

The work with topological quantum computation (TQC, see [E9]) provided very fruitful stimuli and insights about quantum TGD itself. TQC [30, 31, 32] involves topological notions like links, knots and braids, and topological invariants of 3-manifolds. The pioneering works was done by Jones with knot polynomials [35] who noticed also the connection with von Neumann algebras and so called Beraha numbers [36] appearing also in the quantum group context and in representations of braid groups. Topological quantum field theories (TQFT) [33] pioneered by Witten [34] provide an extremely powerful and abstract approach to the deduction of these invariants. The computations of these invariants involve a diagrammatic approach based on recursion bring in mind Feynman diagrams.

There is close relationship with conformal quantum field theories (CQFT) and related mathematical structures referred to as co-algebras, bi-algebras,Hopf algebras,quantum groups and category theory [19]. Since the generalized conformal invariance is a basic element of quantum TGD, this raises the hope that the replacement of "topological" with "conformal" might not not be so drastic a modification after all. If this is the case, one might hope that the generalization of Feynman diagrams might be based on the generalization of braid diagrams by bringing in the interaction in which the strands of braid can decay and fuse.

Even more, one might hope that the notion of equivalence for braid diagrams allowing to transform these diagrams to each other by "elementary moves" could allow to identify loop diagrams with tree diagrams so that divergence problem would disappear.

2.2.2 Fusion rules of conformal field theories and generalization of duality

Conformal invariance is the basic symmetry of also quantum TGD so that it is natural to look what one might learn from conformal quantum field theories, which rely on the notion of field algebra.

In conformal field theories fusion rules [19] for operator algebra code the vertices of the theory to numerical coefficients: $\Phi_k(z,\overline{z})\Phi_l(w,\overline{w}) = C_{kl}^m(z-w,\overline{z}-\overline{w})\Phi_m(w,\overline{w})$ modulo derivative terms which vanish in vacuum expectation values. Conformal invariance dictates two-point functions completely and three-point functions apart from numerical coefficients since one can write $C_{kl}^m(z,\overline{z}) = z^{\Delta_m-\Delta_k-\Delta_l}\overline{z}^{\overline{\Delta}_m-\overline{\Delta}_k-\overline{\Delta}_l}C_{kl}^m$. 4-point functions and also higher n-point functions can be constructed once the coefficients C_{kl}^m are known.

In particular, four-point functions can be expressed in terms of three point functions by interpreting them in terms of particle scattering as a sum of s-channel resonance or t-channel exchanges and the coefficients C_{kl}^m appear explicitly in these conditions. Duality corresponds to crossing symmetry which is of special importance in TGD framework where particle reaction can be interpreted as a creation of a state with vanishing conserved quantum numbers from vacuum. Rather remarkably, crossing symmetry is in turn equivalent with the associativity of the field algebra. One can express four-point function in terms of so called conformal blocks and crossing symmetry leads to conditions which can be solved in many cases.

This occurs when the primary field content of the theory is finite. This occurs when the Verma modules $\mathcal{V}^{\Delta}$ associated with the primary fields contain zero norm states and thus contain a finite number of descendants (being thus effectively finite-dimensional as representations of conformal algebra) and when the operator products of the primary fields close to a finite algebra. The exceptional weights Δ are algebraic numbers involving only square roots of integers and thus p-adically highly interesting. In the case of rational (super-)conformal field theories the conformal weights are rational numbers and conformal Ward identities allow the construction of the descendants of the primary field Φ_{Δ} explicitly. The conformal weights for the finite number primary fields can be listed and fusion coefficients written explicitly. Same applies to super-symmetric variants of these theories. Rational conformal field theories are of special interest in TGD framework since there are good hopes that at least they allow a continuation to p-adic number fields. Note however that the allowed conformal weights for theories allowing null vectors are always algebraic numbers.

It would seem that the field algebra of conformal field theories might closely relate to what happens in 3-particle vertex for incoming particles. Fusion corresponds to an algebra product. In conformal field theories also co-algebras for which multiplication is replaced with co-multiplication analogous to a particle decay are important. This encourages to think that a suitable generalization of Hopf algebra with duality involving both multiplication μ and co-multiplication Δ in a well defined sense time reversals of each other by the duality, should be fundamental for the algebraization of Feynman rules.

String model duality which allows to identify diagrams involving resonances in s-channel with diagrams involving exchanges in t- or u-channel. The additional space-dimension brought in by TGD might mean a generalization of duality so that any diagram with loops would be equivalent with tree diagram, or more generally, to a diagram imbeddable without intersections to a Riemann surface with minimal genus and having minimal number of loops (photon photon scattering suggests this). In the following the consideration is restricted to tree diagrams.

The commuting diagrams expressing the fact that multiplication μ is co-algebra morphism and co-multiplication Δ is algebra morphism indeed state that box diagram describing two-particle scattering is equivalent with tree diagram and it turns out that additional natural postulates would state that also self energy diagrams and vertex correction diagrams are equivalent with corresponding tree diagrams.

2.3 How to end up with generalized Feynman diagrams in TGD framework?

In TGD framework there are several manners to end up with the proposed vision about Feynman diagrams with generalized moves eliminating loops. The idea that loop integrations giving rise to the renormalization of coupling constants and to anomalous dimensions vanish is consistent with this view and also emerges naturally in TGD framework.

2.3.1 Quantum classical correspondence and the replacement of sum over Feynman diagrams with single diagram

The basic difference between TGD and standard QFT:s is that the construction of configuration space geometry assigns to a given 3-surface X^3 a unique space-time surface $X^4(X^3)$ as absolute minimum of Kähler action or some more general preferred extremal [E2]. Although the classical non-determinism of Kähler action forces to modify this picture, the implication is that the functional integral over all 4-surfaces X^4 going through X^3 is replaced with single 3-surface $X^4(X^3)$. Quantum classical correspondence allows the interpretation of $X^4(X^3)$ as a generalized Feynman diagram. Hence single Feynman diagram without loops, "tree diagram", would characterize particle reaction and correspond the simplest possible generalized Feynman diagram among many equivalent ones. The vacuum degeneracy of Kähler action and the consequent non-determinism might mean that there indeed exists a large, or even infinite, number of equivalent absolute preferred Bohr orbit like extremals of Kähler action [B1, E2] equivalent as far as computation of S-matrix is considered. Deterministic regions of space-time sheets could be seen as diagrams with incoming and outgoing on mass shell particles. Also the possibility that these diagrams are not equivalent can be considered. "Homotopically non-equivalent" diagrams could be perhaps interpreted as time evolutions occurring in different quantum phases.

2.3.2 Quantum criticality requires the vanishing of loop corrections

Configuration space integration over quantum fluctuating degrees of freedom leads to a perturbative expansion similar to the ordinary Feynman diagrammatics. Quantum criticality states that the Kähler coupling strength is analogous to a critical temperature and has only a discrete set of values, at least one for each p-adic number field R_p and each algebraic extension of R_p. Quantum criticality requires loop corrections to the configuration space integral vanish so that it would reduce effectively to a Gaussian integral around a maximum of Kähler function for given values of zero modes.

Coupling constant evolution assignable with loop diagrams would be coded by the dependence of "critical temperature" on p (p-adic length scale) and algebraic extension of R_p. The loop gymnastics could be seen as an extremely tedious manner to model discrete p-adic coupling constant evolution having more elegant description in terms of quantum criticality.

2.3.3 p-Adicization requires the vanishing of loop corrections

Also the hypothesis that S-matrix elements can be algebraically continued to real and various p-adic number fields encourages the hope that configuration space integration must effectively reduce to an effectively Gaussian integral around a maximum of Kähler function so that simple expressions involving only rational, algebraic, and exponential functions requiring a finite-dimensional extension of p-adic numbers result making it possible to continue the expressions to p-adic number fields. The reason is that infinite sums of diagrams are not expected to give a result boiling down something expressible in terms of these functions.

Theory would formally reduce to a free field theory and the non-triviality of the theory would be basically due to the topological description of particle reactions. The symmetric space property of the quantum fluctuating degrees freedom and Duistermaat-Hecke theorem [37] stating that functional integral can be expressed as an exponential of Kähler function K associated with the maximum of K give indeed good hopes that the dream might be realized.

2.3.4 The construction of S-matrix as analytic continuation between different sectors of configuration space

By quantum classical correspondence particle reactions correspond to processes changing 3-topology. For instance, the decay of a particle to two final state particles in Fock sense corresponds to a decay of a connected 3-surface to two disjoint components. In string models this can be described using smooth string world sheet whereas the string representing the vertex is singular eye glass like configuration. TGD suggests a different approach in which lines are replaced with singular four-manifolds whose ends meet at non-singular 3-manifold representing vertex is more appropriate. As a matter fact, the first proposal to describe S-matrix in terms of CP_2 type extremals led to this picture but was thought to be a mere approximation at that time [C3]. In this picture the stringy branching of the 3-surface would have interpretation as a space-time correlate for what it means that particle traverses simultaneously through several paths as in the case of double slit experiment. This certainly conforms with what happens to the induced second quantized spinor field in the branching.

For both options the generalized Feynman diagrams could be seen as characterizing the paths involved with a continuation of a configuration space spinor field from a given sector to another one. Any path will do and this implies a large number of moves allowing to construct the simplest possible Feynman diagram characterizing the continuation. Each topology change would give rise to a vertex and motion inside given sector would presumably correspond to a stringy propagator. Also other than topological characteristics, say the parameters characterizing conformal structure, might be involved.

At the space-time level the continuation would reduce to a continuation of second quantized free induced spinor fields from a given 3-topology to a new one. The 3-vertices describing decay and fusion provide the basic examples. The task of continuing an induced spinor field and corresponding fermionic Fock space from single-particle sector to two-particle sector means finding of an algebra morphism imbedding a single particle Fock algebra to a tensor product of Fock algebras and is highly analogous to finding of a co-algebra product Δ. The time reversal of Δ corresponds to a fusion vertex at the Fock space level. The construction of S-matrix would reduce to the solution of this continuation problem and one might hope that the process boils down to general algebra axioms.

2.3.5 The notion of Platonia and Feynman diagrams as computations

As found in [E10], the notion of infinite primes [E3] leads to a generalization of real numbers since an infinite algebra of real units becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

The along curve makes means that both real and p-adic norms of the unit multiplying the point along curve are constant. The curve allows interpretation as a representation of an energy conserving time evolution of an arithmetic quantum field theory. In a complete analogy with the continuation at the configuration space level and analytic continuation, the time evolution defined by the cobordism does not depend on the path traversed. Each curve represents a discrete sequence of phase transitions changing the unit and the interpretation as a Feynman diagram like structure is very attractive.

The requirement that the configuration space paths defining generalized Feynman graphs are cognitively representable as cobordisms of a point provides a powerful heuristic tool in the attempts to say something non-trivial about the general properties of S-matrix. The cobordism of a point can be generalized in several manners.

In this approach Feynman diagrams could be seen as sequences of computations satisfying some algebraic rules fixed by the model that is mimicked. Computations themselves form a local algebra. The basic super rule satisfied always would be the conservation of guaranteing real and p-adic continuities. The generalized Feynman diagram defining a representation of an algebraic computation would determine an element of algebra since the algebra elements associated with different points of space-time surface form a local algebra structure. The simplest possible Feynman diagram would correspond to a tree diagram.

Many interpretations are possible. The preferred Bohr orbit like extremals of Kähler action could represent the simplest computations leading from an input to the outcome or minimal sequences of steps making up a proof of a theorem. At the level of configuration space, the simplest Feynman diagram would correspond to the simplest path allowing to continue configuration space spinor field from a given sector D_1 with a fixed 3-topology to another sector D_2.

The algebraic interpretation implies powerful symmetries. By replacing the topological invariance with the generalized conformal invariance serving as the basic symmetry of also TGD, it might be possible to generalize the construction of modular invariant S-matrices of topological quantum field theories to that of conformally invariant S-matrices by adding the fusion and decay of braid strands as additional operations to an appropriate braided category. The very intimate relationship between topological and conformal quantum field theories raises the hope about concrete predictions.

3 Algebraic physics, the two conformal symmetries, and Yang Baxter equations

TGD predicts two conformal symmetries corresponding to super-canonical symmetries acting at the level of imbedding space $H = M^4 \times CP_2$ and Super-Kac-Moody symmetries acting at the space-time level. If the notion of number theoretic spontaneous compactification makes sense there are also the analogs of conformal symmetries defined by hyper-octonion analytic maps of $OH = M^8$ and hyper-quaternion analytic maps of space-time surface regarded as a hyper-quaternionic 4-surface in M^8 [E2]. These symmetries would induce dynamical symmetries at the level of H.

1. The super-canonical conformal algebra appears naturally in the construction of the configuration space geometry for $H = M^4_+ \times CP_2$ option at the 7-dimensional light-like boundary of $M^4_+ \times CP_2$ and has complex conformal weights, possibly identifiable as zeros of Riemann zeta. The non-determinism of Kähler action and aesthetic considerations encourage the expectation that all light-like surfaces X^3_l of M^4 define 7-dimensional light-like causal determinants $X^3_l \times CP_2$ in this sense. Light-like 3-surfaces form a foliation of M^4 or M^4 by 4-dimensional general coordinate invariance and one can use any light-like 3-surface. The classical non-determinism of course poses restrictions.

2. The super-conformal algebra associated with the 3-dimensional light-likes surfaces X^3_l, which can appear as boundaries of space-time sheets and as elementary particle horizons at which the induced metric is degenerate corresponds to the ordinary super Kac-Moody algebra SKM of elementary particle physics and of string models. Conformal weights are now real and half integer valued.

The challenge is to understand how these symmetries relate to each other and and how they interact. Quantum classical correspondence allows to deduce a rather detailed interpretation of the basis of super-canonical algebra interpreted as super-conformal fields in the space defined by the complex conformal weights labelling them, and SKM conformal algebra having natural infinitesimal action in the space of super-canonical weights. Therefore he powerful machinery of super-conformal theories becomes directly available and allows to make educated guesses about the basic structure of S-matrix.

Yang-Baxter equations characterize the physics of two-dimensional integrable systems [18, 19] and define a core element of braided Hopf algebras, braid group representations appearing in topological and conformal quantum field theories. R-matrices relate very directly to the Clifford algebra of the configuration space gamma matrices defining the fermionic part of super-canonical algebra and they act at the space-level too. Hence the physics of two-dimensional integrable systems and conformal field theories become a quintessential part of quantum TGD.

3.1 Space-time sheets as maximal associative sub-manifolds of the imbedding space with hyper-octonion structure

The vision about physics as a generalized number theory stimulated a development, which led to the notion of number theoretic compactification [E2]. Space-time surfaces can be regarded either as hyper-quaternionic, and thus maximal associative, 4-surfaces in M^8 having hyper-octonion structure, or as surfaces in $M^4 \times CP_2$ [E2]. Hyper-quaternions/-octonions form a subspace of complexified quaternions/-octonions for which imaginary units are multiplied by $\sqrt{-1}$: they are needed in order to have a number theoretic norm with Minkowski signature. What makes this duality possible is that CP_2 parameterizes different quaternionic planes of octonion space containing a fixed imaginary unit. S^2 can in turn be interpreted as the parameter space labelling the maximal commutative subspaces, i.e. complex planes, of a quaternionic space. The specification of quaternion plane is necessary in order to introduce octonion Hermitian structure whereas the specification of preferred imaginary unit (that is complex plane) is necessary in order to fix Hermitian structure in hyper-quaternionic tangent plane [E2].

Hyper-quaternionic sub-spaces are also maximal associative sub-spaces. The physical counterpart of associativity is crossing symmetry: in conformal field theories crossing symmetry follows from the associativity for the products of fields. The Yang-Baxter equations emerging naturally in 2-dimensional integrable quantum field theories and statistical systems state the associativity of braiding operations in the case of three strand braid. Thus it would not be surprising if there were connection between Yang-Baxter equations and space CP_2 of quaternionic sub-spaces of octonions. Commutativity of R-matrices means commutativity of braiding operations and this suggests that also the sphere labelling complex planes of quaternion space might emerge naturally in the context of Yang-Baxter equations.

Indeed, the Yang-Baxter matrices defining equivalent 6-vertex models are parameterized by CP_2 whereas mutually commuting Yang-Baxter models by sphere S^2. These findings lead to a quite detailed albeit speculative view about quantum classical correspondence.

1. One can select a 4-parameter subset of equivalent R-matrices

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \tag{1}$$

   in such a manner that they are apart from a real multiplicative constant r unitary matrices labelled by 3-parameters (Θ, Φ, Ψ) and one has

$$R = rexp(i\Phi) \times \begin{pmatrix} exp(i\Psi) & 0 & 0 & 0 \\ 0 & cos(\theta) & isin(\theta) & 0 \\ 0 & isin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & exp(i\Psi) \end{pmatrix} \tag{2}$$

   r is identifiable as the $U(2)$ invariant radial coordinate of CP_2 and (Θ, Φ, Ψ) are identifiable as spherical coordinates of $r = constant$ 3-sphere of CP_2.

   If it is possible to interpret the points of CP_2 as labels of R-matrices, one could assign to a given point of space-time surface a unitary R-matrix labelling the quaternion structure of the tangent space at that point. The R-matrices with the same value of the parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} \tag{3}$$

   commute as is easy to verify. With the parametrization used this means that the commuting R-matrices corresponds to the sphere defined by the equation

$$\frac{cos(\theta)}{cos(\psi)} = constant \ . \tag{4}$$

2. A classical space-time correlate for equal time commutation relations of quantum field theories suggests itself. In the generic case the space-time surface would decompose into lines along which r varies and the "S-matrix" defined by the R-matrix would be constant along these lines. One could choose the time coordinate in such a manner that time constant 3-surface would foliated this kind of lines. The R-matrices associated with a 3-surface would be labelled by two coordinates, and by a proper selection of time coordinate it might be possible to decompose time=constant 3-surfaces to regions inside which R-matrices commute with each other and correspond to different complex planes in the space of quaternions. The R-matrices corresponding to different values of time coordinate would not commute.

3. Space-time evolution would induce an adiabatic evolution of a conformal quantum field theory. In particular, an evolution of the transfer matrix or S-matrix of a two-dimensional integrable model with respect to the parameters characterizing the matrix, and of n-point functions of a conformal field theory would be induced.

4. Particle decay could be interpreted in terms of a flow in which 3-surface X^3 decomposes to two disjoint components Y^3 and Z^3 such that the R matrices associated with different points y of Y^3 *resp.* z of Z^3 commute but the commutativity fails for R-matrices associated with y and z. The loss of commutativity and decay to pieces can be interpreted in terms of quantum de-coherence. This flow interpretation might significantly help in the construction of mathematical description of particle reactions.

   Of course, here one encounters two possible interpretations for what particle decay means corresponding to the stringy picture and vertex identified as a 3-surfaces which defines the common end of incoming 4-surfaces. If both 3-surfaces decompose to union of one-dimensional curves along which the S-matrix defined by R-matrix is constant, the stringy decay could be illustrated graphically also as a decay of string. For second option a replication of string would be in question.

3.2 Super Kac-Moody and corresponding conformal symmetries act on the space of super-canonical conformal weights

TGD predicts to super-conformal algebras. Super-canonical algebra certainly appears in the construction of the configuration space metric and spinor structure. Quaternion conformal algebra which generalizes the Kac Moody algebras of string models and its role is somewhat unclear in this respect.

3.2.1 Single particle super-canonical conformal weights and zeros of Riemann Zeta

Configuration space spinors can be interpreted as systems describing Fock states consisting of fermions with spin, electro-weak and color quantum numbers, plus a complex conformal weight. The work with conformal invariance and Riemann hypothesis (see [E8] and [16, 17]) inspires the hypothesis that the complex conformal weights labelling configuration space gamma matrices and isometry generators defining super-canonical algebra could be determined by the zeros of Riemann Zeta.

The conformal weights of the physical states would correspond to $Re(h) = n - 1/2 - i\sum n_i y_i$, $n \geq 0$ or $n = 0, 1$ at least, and to real axis the points $h = 2n$, $n > 0$, corresponding to trivial zeros at $s = -2n$. The superpositions $\sum n_i y_i$ of the imaginary parts of trivial zeros $s = 1/2 + y_i$ must be allowed. These points label the algebra elements obtained by commuting the algebra generators labelled by conformal weights which are the negatives of the zeros of Riemann Zeta and negative integers. If one assumes that Virasoro generators L_n, $n \geq 1$, generate zero norm states, only the $n = 0, 1$ option for the complex conformal weights remains. This option is forced also by the construction of the scalar propagator as a partition function in the super-canonical algebra. $n = 0, 1$ restriction guarantees also the orthogonality of the physical states.

These 1-dimensional lattices could define braid groups possessing an infinite number of generators appearing also in the description of type II_1 factors of von Neumann algebras [20] as described in [E10]. The Clifford algebra of the configuration space spinors would be the natural identification for von Neumann algebra in question. The connection between hyperfinite type II_1 factors of von Neumann algebras, braid groups, and quantum groups suggests that the existence of a quantum group structure at least in configuration space spinor degrees of freedom. Also the braiding for ordinary conformal field theories is known to define quantum group structure in a natural manner. The connection between R-matrices and quaternionic and complex structures supports this expectation. The minimal assumption would be that configuration space Clifford algebra for a given three-topology can be regarded as ordinary algebra without co-product and that co-product structure is associated only with the continuation of the Fock space structure from single-particle sector to two-particle sector.

Configuration space gamma matrices defining super generators and corresponding bosonic generators could be interpreted as conformal fields in the complex plane restricted to the punctures so that the vacuum expectation values of Clifford algebra elements would have interpretation as correlation functions of a conformal quantum field theory evaluated at points which define the lattice of super-canonical conformal weights. Also a connection with integrable lattice models suggests itself. In field theory context one possible interpretation is that a particle of given mass has an infinite degeneracy corresponding to the various super-canonical conformal weights and that one must sum in Feynmann diagrams over this degeneracy somehow. Physical intuition suggests wave functions in the set of conformal weights. This point will be discussed in detail later.

3.2.2 Mapping of super-canonical conformal weights to a geodesic sphere of CP_2

By quantum classical correspondence the construction of the S-matrix in configuration space spinor degrees of freedom should reduce to a corresponding construction at the space-time level. The idea that real physics and various p-adic physics result through an algebraic continuation of rational physics does not force discretization at the space-time level but leads in a natural manner to emergence of discrete sets of rational points as intersections of real and p-adic space-time sheets [E1]. Both infrared and ultraviolet cutoffs are involved since p-adically infinitesimal is infinite in the real sense and vice versa (for some values of p for given real infinitesimal) so that most points of p-adic space-time sheets are at real infinity. Each discretization corresponds to a different extension of p-adic numbers such that an improving resolution means increasing dimension of extension implying improved cognitive resolution.

The second quantized free spinor fields with a discretization at the space-time level could be interpreted in terms of a conformal field theory. Quantum classical correspondence would mean the mapping of the super-canonical conformal weights to the space-time points, which correspond to points of geodesic sphere of CP_2 giving rise to a family of commuting R-matrices. The mapping of complex scaling momenta to complex surface S^2 of CP_2 is analogous to the mapping of ordinary momenta to the points of the celestial sphere relating quantum to classical. This mapping is natural because configuration space gamma matrices are linearly related to the second quantized induced spinor fields. The physical states in configuration spin degrees of freedom might be characterized by the states of this system. In the real context this would require taking the limit of vanishing UV cutoff.

Braiding operation characterized by the Yang-Baxter matrix R for a tensor product of identical representations of braid group is the key element of braided Hopf algebras and the considerations of [E10] led to the speculation that braiding operation acts in the Clifford algebra of configuration space. The light like boundaries of space-time surfaces allow generalized conformal structure as metrically 2-dimensional structures. In a light hearted manner I christened this conformal symmetry as quaternion conformal invariance (the punishment has been a colossal find and replace procedure). A better term would have been Super Kac-Moody symmetry. Hyper-quaternionic and -octonionic analogs of conformal symmetry emerge naturally if the notion of the number theoretic spontaneous compactification makes sense.

The braiding operation for "particles" located in 2-dimensional sections could be seen as a space-time correlate for a corresponding operation performed for configuration space gamma matrices. The complex conformal weights of configuration space gamma matrices for which also imaginary part should be discrete in order to allow p-adicization could be interpreted as counterparts for the coordinates of points of a one-dimensional lattice defining an integrable system characterized by R-matrix. Configuration space gamma matrices would become conformal fields as function of super-canonical conformal weight on which super Kac-Moody conformal symmetries act infinitesimally.

3.2.3 Some ideas about generalized coset construction

In the following some ideas about generalized coset construction involving super-canonical and super Kac-Moody algebras are discussed. Unfortunately, this discussion has become somewhat obsolete and is not in accordance with the recent view about generalized coset construction discussed in [C1] and relying on the notion of zero energy ontology.

1. Generalization of the coset construction as a manner to satisfy Super Virasoro conditions

Super Virasoro constraints pose an extremely powerful constraint on theories possessing conformal invariance. The recent view about particle massivation [F2] inspires the hypothesis that Olive-Kent-Goddard coset construction [48]generalizes in such a manner that the sums of super-canonical and Super Kac Moody Super Virasoro generators annihilate the physical states. This would provide a very elegant manner to understand and realized super conformal invariance.

A possible interpretation would be in terms of a duality stating that these two super algebras correspond to two different coordinatizations of the physical degrees of freedom of the configuration space and configuration space spinors using imbedding space vector fields *resp.* imbedding space vector valued fields defined at the space-time surface. In this picture one has also hopes of understanding why the conformal transformations associated with Super-Kac Moody algebra affect the conformal weights of super-canonical algebra.

2. Constraints from unitarity

For the representations of Virasoro algebra allowing Verma modules containing null states annihilated by Virasoro generators L_n, $n > 0$ conformal weights are quantized [19]. The simplest situation would correspond to rational conformal weights Δ and maximally symmetric superconformal quantum field theories known as rational super-conformal field theories (RCFT) having a finite number of primary fields. In TGD framework super-generators carry fermion number which leaves $N = 2$ super conformal theories (or possibly $N = 4$ theories, situation is still unsettled) as the only physically acceptable option. The conformal weights are predicted to be rational numbers in this case.

For Sugawara construction [19] the super Kac-Moody conformal weights associated with the Hamiltonians are expressible in terms of the value of Kac-Moody central extension parameter k and Casimir invariant of the highest weight representation. For instance, in the case of $SU(2)$ one has $\Delta(j) = \frac{j(j+1)}{k+2}$. Unitarity poses strong constraints on the super Kac-Moody conformal weights.

In Wess-Zumino-Witten model also the analogs of Verma modules having finite number of non-zero norm states for Kac Moody algebras emerge and relate very closely to the corresponding phenomenon for quantum groups [19]. Thus only a finite number of highest weight representations of Kac Moody algebra appear as primary fields in WZW model. The theory predicts the dependence of the conformal weights on the finite-dimensional representations of the Kac Moody algebra. In the case of super Kac-Moody conformal algebra the super Kac-Moody conformal Kac-Moody representations relate naturally also the finite dimensional representations of $SO(3) \times U_{ew}(2) \times SU(3)$ labelling finite-dimensional or highest weight Kac Moody representations.

For Sugawara construction central extension parameter is given by $c = kdimg(G)/(k+h^{\star})$, where $h^{\star}$ is the dual Coxeter number ($h^{\star} = n$ for $SU(n)$ and is non-vanishing in general case and one faces the question whether super Kac-Moody conformal invariance is gauge symmetry or not. String model based mass formulas would suggest the super Kac-Moody Virasoro representations should correspond to a trivial central extension $c = 0$.

The most general hypothesis is that all values of k and c consistent with unitarity are possible. Elementary particles would correspond to $k = 0, c = 0$ and finite-dimensional representations defined by Hamiltonians belonging to a given representation of $SO(3) \times SU(3)$. Anyons could correspond to a non-vanishing value of k. At configuration space level this would correspond to an extension of isometry generators obtained by adding an integer multiple of Hamiltonian as a scalar term to the isometry generator so that the action couples together several irreducible representations of $SO(3) \times SU(3)$ corresponding to states with non-vanishing norm.

3.3 Could Super Kac-Moody and conformal algebras act on radial super-canonical conformal weights as infinitesimal conformal transformations?

Quantum classical correspondence inspires the hypothesis that super Kac-Moody algebra and corresponding Virasoro algebra act somehow on the complex conformal weights Δ assignable to the functions $(r_M/r_0)^{\Delta}$ of the radial coordinate r_M of $\delta M^4_{\pm}$ in super-canonical algebra consisting of functions in $\delta M^4_{\pm} \times CP_2$. The following arguments suggest a detailed realization of this misty idea and provide also a deeper reason why for it.

3.3.1 Basic argument

The basic argument runs as follows.

1. Let us start from the idea that the discrete set of points of the geodesic sphere S^2 of CP_2 labelling commuting R-matrices should correspond to the super-canonical conformal weights Δ assignable to the functions $(r_M/r_0)^{\Delta}$ of the radial light-like coordinate r_M of $\delta M^4_{\pm}$ in super-canonical Hamiltonians.

2. This discrete set of conformal weights should correspond to a discrete set of points at the partonic 2-surface X^2 defined as the intersection of 3-D light-like causal determinant X_l^3 defining the orbit of parton and $\delta M_\pm^4 \times CP_2$. This selection of a discrete subset of points in the CP_2 projection of X^2 would make it possible to realize radial conformal weights as points of a "heavenly sphere" defined by the CP_2 projection. A homologically non-trivial geodesic sphere $S^2 \subset CP_2$ would provide a natural complex coordinate for the projection with S^2 isometries acting as Möbius transformations for the preferred complex coordinate z shared by X^2 and $S^2 \subset CP_2$. This assumption is however unnecessarily strong as will be found.

3. The finite set of points having interpretation as a braid would belong to a "time=constant" section of 2-dimensional "space-time", presumably circle, defining physical states of a two-dimensional conformal field theory for which the scaling operator L_0 takes the role of Hamiltonian.

3.3.2 Radial conformal weight as a function of CP_2 coordinates

One can criticize the idea of assigning radial conformal weights with CP_2 points as an ad hoc procedure. Nothing however prevents of modifying the original definition of Hamiltonians of $\delta M_\pm^4 \times CP_2$.

1. One could *redefine* the configuration space Hamiltonians by assuming that radial conformal weights Δ are functions of CP_2 coordinates so that the radial parts of Hamiltonians would be of form $(r_M/r_0)^{\Delta(s)}$.

2. For instance, one could have $\Delta = \zeta^{-1}(\xi^1/\xi^2)$, where ξ^i are complex CP_2 coordinates transforming linearly under group $U(2)$: CP_2 itself parameterizes these coordinate choices. For a given value of $r = r_M/r_0$ one should select some branch of the inverse of ζ in this formula. Branches are in one-one correspondence with zeros of Zeta and they could be in one-one correspondence with partonic 2-surfaces assignable to a given 3-surface.

3. One can worry about the non-uniqueness of the selection of preferred CP_2 coordinates. As a matter fact, in its recent form the construction of configuration space as a union of sub-configuration spaces involves, not only a selection of the tip of the light cone of M^4, but also a selection of a preferred point of CP_2 so that the points of H label sub-configuration spaces [C8, C1]. The physical interpretation is in

terms of a geometric correlate for the selection of Cartan algebra of commuting color charges.

4. The requirement that the Hamiltonian belongs to an algebraic extension of p-adic numbers forced by the p-adicization constraint would select a discrete set of points of X^2. Thus the discrete spectrum of conformal weights expressible in terms of zeros of Riemann Zeta would result from the number theoretical quantization forced by the p-adicization constraint.

3.3.3 Kac-Moody generators in CP_2 degrees of freedom and Virasoro generators have a natural action on radial conformal weights

Kac-Moody generators acting in CP_2 degrees of freedom as X^2-local color rotations would act on the conformal weights identified as points of CP_2. Also Virasoro generators have a similar action.

1. If the projection of X^2 to CP_2 is 2-dimensional, the complex coordinate of X^2 can be identified as CP_2 coordinate so that also conformal transformations would induce an action in CP_2 degrees of freedom and thus on radial conformal weights.

2. The fact that radial conformal weights define a discrete set on a 1-dimensional curve, allows the identification with CP_2 points even when the CP_2 projection is 1-dimensional and one can restrict conformal transformations to this line.

3. Since the number of the radial conformal weights is discrete, it is possible to compensate for the action of the local Kac-Moody generator on conformal weights by an infinitesimal conformal transformation expressible as a superposition of Virasoro generators. Both CP_2 Kac-Moody algebra and conformal algebra would act on super-canonical conformal weights and induce braiding operations naturally.

3.3.4 Why a discrete set of points of partonic 2-surface must be selected?

As already noticed, p-adicization might provide a deeper motivation for the selection of discrete subset of points of partonic 2-surface in the construction of S-matrix elements.

1. As shown in [C2], the fusion of p-adic variants of TGD with real TGD, could be possible by algebraic continuation. This however requires the restriction of n-point functions to a finite set of algebraic points of X^2 with the usual stringy formula formula for S-matrix elements involving an integral over a circle of X^2 replaced with a sum over these points.

2. The same universal formula would give not only ordinary S-matrix elements but also those for p-adic-to-real transitions describing transformation intentions to actions. Quite generally, the formula would express S-matrix elements for transitions between two arbitrary number fields as algebraic numbers so that p-adicization of the theory would become trivial.

3. The interpretation of this finite set of points as braid conforms also with the fact that the hierarchy of Jones inclusions which defines a fundamental piece of quantum TGD in its recent form corresponds to a sequence of inclusions for braids [C7]. What the precise physical interpretation of this braid hierarchy correlating with the hierarchy of algebraic extensions of p-adic numbers is, remains to be understood.

What could then be this discrete set of points having interpretation as a braid?

1. Number theoretical vision suggests that quantum TGD involves the sequence hyper-octonions $\rightarrow$ hyper-quaternions$\rightarrow$ complex numbers$\rightarrow$ reals $\rightarrow$ finite field $G(p,1)$ or of its algebraic extension. These reductions would define number theoretical counterparts of dimensional reductions. The points in the finite field $G(p,1)$ could be defined by p-adic integers modulo p so that a connection with p-adic numbers would emerge. Also more general algebraic extensions of p-adic numbers are allowed.

2. If the exponents p^{iy} for the zeros $z = 1/2 + iy$ of Zeta are algebraic numbers, the linear combinations for a finite subset of them for a given algebraic extension of p-adic numbers would naturally represent preferred points of $S^2 \subset CP_2$ using the group-theoretically preferred complex coordinate z of S^2. Note that radial coordinate which itself is expressible in terms of CP_2 coordinates when partonic 2-surface has 2-dimensional CP_2 projection, must be rational which poses also conditions on the allowed set of points.

3. If the radial conformal weights are expressible in terms of CP_2 coordinates, such that for a restriction to $S^2 \subset CP_2$ one has $\Delta = \zeta^{-1}(z = \xi^1/\xi^2)$, the basic condition would be that $r = r_M/r_0$ is rational and r^Δ belongs to the algebraic extension of p-adic numbers used. The rationality of $r_M(z)$ as function of S^2 coordinate z would select a subset of linear combinations of zeros of Zeta.

3.3.5 What is the fundamental braiding operation?

5. What is the fundamental braiding operation?

The basic quantum dynamics of TGD could define the braiding operation for the braid defined by a discrete set of points of X^2 satisfying the algebraicity conditions.

1. The complex coordinate z of X^2 is defined by the conformal equivalence class of the induced metric of X^2 apart from conformal transformation and positions of punctures are expressed using this coordinate.

2. The selection of the radial coordinate r_M defines a rest system and thus a foliation of $M^4_\pm$ by light-cone boundaries parameterized by M^4 time coordinate m^0 giving the temporal location of the tip of $\delta M^4_\pm$ along the line $r_M = 0$. This foliation defines also a foliation of the partonic orbit defined by the light-like 3-surface X^3_l by partonic 2-surfaces $X^2(m^0)$. Each of these surfaces defines a number theoretic braid and one expects that the positions of the punctures of braid expressed using coordinate z evolve most of the time in a continuous manner so that the braiding flow is well-defined. If one has $\Delta = \zeta^{-1}(z_1 = \xi^1/\xi^2)$ then z_1 remains constant during the flow and only z changes meaning that M^4 projection corresponding to z_1 changes. Hence, if M^4 projection is 2-dimensional, the braiding flow can be regarded as a flow defined at M^4 projection, which of course conforms with the physical picture.

3. New points can appear to and old points disappear from the braid so that a structure more general than a mere braid is in question. It would not be surprising if the points satisfying algebraicity conditions disappear or are created in pairs. If this is the case, a tangle like structure allowing topological particle pair creation and annihilation would be in question.

This picture makes sense also for macroscopic 2-surfaces defining outer boundaries of physical systems (quantum Hall effect and topological quantum computation [E9]).

3.4 Stringy diagrammatics and quantum classical correspondence

One expects that S-matrix can be constructed by generalizing the ordinary Feynman diagrams in the manner already discussed whereas generalization of stringy tree diagrams would have different interpretation in TGD framework. This expectation combined with quantum classical correspondence has some interesting implications.

3.4.1 Vertex operators and configuration space super algebra

Vertex operators are a fundamental notion in conformal field theories and string models. Quantum classical correspondence suggests that S-matrix should be expressible in terms of the vertex operators which at the configuration space level correspond to the continuations between one- and two-particle sectors of configuration space. By quantum classical correspondence the conformal weights of the generators of super-canonical algebra generators at X^3 are mapped to complex points of a two-sphere of CP_2 in turn defining points of X^3 and interpreted as punctures of a compactified complex plane. This picture conforms with the fact that sewing procedure allows to construct n-point functions for arbitrary 2-topology and with an arbitrary number of punctures using spheres with 3 punctures as a basic building block.

One can understand why the number of punctures appearing in vertex function cannot be larger than three. Three punctures means three complex super-canonical conformal weights and globally defined conformal transformations acting on the complex sphere must map this set of punctures to itself. The group $SL(2,C)$ of Möbius transformations acts as global conformal symmetries of sphere and can indeed map these points to each other. If the number of points is larger than 3 this is not possible unless the points satisfy some special symmetries.

Conformal invariance fixes the dependence of 3-vertices $\langle\Phi(z_1)\Phi(z_2)\Phi(z_3)\rangle$ on the coordinates z_i, which now correspond to super-canonical conformal weights. What remains is the possibility that the fusion coefficients C^k_{lm} appearing in both the product and co-product depend on two light-like coordinates of the space-time sheet complementary to the complex coordinates

$(w, \overline{w})$. The dependence on CP_2 coordinates labelling different braiding matrices suggests itself as an additional dependence. One might hope that the construction of vertices might not differ much from that in conformal field theories and string models. The most optimistic expectation is that the theory effectively reduces to a construction of stringy diagrams without loops.

3.4.2 Delocalization in the space of super-canonical conformal weights as localization in the imbedding space

The identification of conformal weights as punctures raises the question how to interpret the propagators $1/L_0$ appearing in stringy diagrams. The propagating states are eigen states of the scaling momentum L_0 associated with super Kac-Moody conformal algebra. Therefore the physical states should be quantum superpositions of states with various complex super-canonical conformal weights defining two one-dimensional sub-lattices along lines $Re(s) = \pm 1/2$ and $(Re(s) < 0, Im(s) = 0)$. Scaling momentum eigen states represent Bloch waves in the lattice of punctures analogous to Bloch waves in a one-dimensional lattice encountered for Bethe ansatz [19] for integrable lattice models and spin chain models. Bethe ansatz should apply also now to the construction of states created by configuration space gamma matrices as eigen-states of super Kac-Moody conformal Virasoro generator $exp(iaL_0)$.

The construction of scalar field propagator as a partition function for the physical states of super-canonical algebra demonstrates that the introduction of a hierarchy of y-cutoffs is necessary defined by sub-algebras having as conformal weights the lattice defined by trivial zeros of zeta and n first nontrivial zeros $s = 1/2 + iy_i$, $i = 1, ..., n$, of Riemann Zeta. This cutoff hierarchy is probably related to the hierarchy of type II_1 factors of von Neumann algebras.

In the construction of super-canonical algebra ordinary translations are replaced by scalings acting on the radial light-like coordinate r associated with a light-like 3-surface X_l^3 of M^4.

1. The construction of the super-canonical algebra discussed in [B2] demonstrates also trivial zeros of Zeta correspond naturally to a super-canonical representation and the generators associated with trivial zeros span together with the generators associated with the non-trivial zeros an orthogonal basis. As a matter fact, non-trivial zeros correspond to $SO(3)$- and trivial zeros to $SO(2)$ reduction for the represen-

tations of the Lorentz group and the appearance of only even integers corresponds physically to the fact that only even parity excitations can be assigned to a given particle [B2].

Orthogonality allows also superposition for the imaginary parts of non-trivial zeros implied by the algebra structure. The most general conformal weights are of form
i) $s = 2n$, $n > 0$, and
ii) $s = n - 1/2 - \sum_i n_i y_i$, $n \geq 0$, with $\sum_i n_i = 2N + 1$ for even n and $\sum_i n_i = 2N$ for odd n. The states of type ii) with $n = 0, 1$ form an orthogonal basis but it is not clear whether orthogonality allows all values of n. The states of type ii) having $n > 1$ would naturally correspond to gauge degrees of freedom obtained via the action of the Virasoro algebra generators L_n, $n \geq 1$. The detailed properties of the super-canonical generators with weights $s = 2n$, $n > 0$, discussed in [B2, B3] guarantee that they indeed define non-gauge degrees of freedom.

2. The model for the renormalization group evolution of the scalar field propagator to be discussed later requires the inclusion of both kinds of zeros. The least singular propagator results if only the orthogonal $n = 0, 1$ states of type ii) are included.

3. p-Adicization requires a scaling momentum cutoff so that super-canonical operators appearing in a given length scale have conformal weights above some resolution. Hence one could have a finite lattice determined by the selection of a lattice generated by a subset of non-trivial zeros of ζ forced by the necessity of pinary cutoff.

4. The appearance of a one-dimensional lattices would be a direct counterpart for the one-dimensional lattice of strands defining the infinite braid characterizing sub-factors of type II_1 for von Neumann algebra. The infinite braids of super-canonical conformal weights at lines $Re(s) = n - 1/2$ ($n = 0, 1$ at least) defined by the points $s = n - 1/2 - \sum_i n_i y_i$, where y_i denote of the first K non-trivial zeros of Riemann Zeta, would correspond to different sub-factor hierarchies for von Neumann algebras. At the space-time level the hierarchies would correspond to different angular (or phase) resolutions and the appearance of the effective non-integer dimensions given by Beraha numbers labelling unitary representations of Temperley Lieb algebra would correspond to anomalous dimensions also in space-time sense [E10].

Quantum classical correspondence implies a strange duality in which the localization at the level of configuration space implies delocalization at the level of space-time. More explicitly, since the eigen-states of the super-canonical scaling momentum are totally delocalized in the radial coordinate r, the Bloch wave localizes the state with respect to r and thus both in imbedding and configuration configuration space. Physically the localization in the scale of 3-surface containing particles is natural. Inside the 3-surface containing particles however delocalization occurs since for the induced spinor fields plane waves over punctures mean de-localization at the level of 3-surface. This is natural since the modes of induced spinor fields are fixed by the requirement that they are eigen-states of L_0.

3.4.3 The zeros of Riemann polyzetas as super-canonical conformal weights of bound or virtual states?

Drinfeld's associator Φ encountered in monodromy considerations of conformal field theories [18] is expressible in terms of Riemann Zeta and polyzetas $\zeta(z_1, ..., z_n)$ [21, 22], and this observation [E10] led to a number theoretical conjecture about the values of ζ and polyzetas at integer valued points (for the role of polyzetas are discussed in a wider context in [28]).

1. Definition of Polyzetas

Riemann's polyzeta $\zeta(z_1, ..., z_n)$ is defined via the sum

$$\zeta(z_1, ..., z_k) = \sum_{n_1 > n_2 > ... > n_k \geq 1} \prod_i n_i^{-z_i} \tag{5}$$

A possible interpretation for the ordering of the integers is in terms of fermion statistics in the sense that symmetrized polyzetas could represent a partition function for a system with states labelled by integers. Already this observation suggests that polyzetas might play some role in physics.

Polyzetas satisfy identities following from their defining representations: how one can deduced these identities from quantum field theory is discussed in [22]. For instance, the identity

$$P_2(a, b) \equiv \zeta(a, b) + \zeta(b, a) = \zeta(a)\zeta(b) - \zeta(a + b) \tag{6}$$

holds true. An example of a more complex identity holding true for 3-zeta is

$P_3(a_1, a_2, a_3) \equiv \sum_P \zeta(a_{P(1)}, a_{P(2)}, a_{P(3)})$
$=$
$$2\zeta(a_1 + a_2 + a_3) + \zeta(a_1)P_2(a_2, a_3) + \zeta(a_2)P_2(a_3, a_1) + \zeta(a_3)P_2(a_1, a_2) \quad (7)$$
$-2\zeta(a_1)\zeta(a_2)\zeta(a_3)$.

Here the subscript P refers to permutations of three objects.

2. Polyzetas and divergences of quantum field theories

Polyzetas $\zeta(k_1, ..., k_n)$ with integer values are found to appear as coefficients of counter terms in quantum field theories. Furthermore, Connes and Kreimer who worked with the renormalization in quantum field theory found that the "forest formula" of Zimmermann can be translated into the language of Hopf algebras and that the counter terms are proportional to the values $\zeta(k_1, ..., k_m)$ [45].

The question discussed Broadhurst and Kreimer [23] concerns the number of polyzeta values $\zeta(k_1, ...k_m)$ for a given weight $\sum_1^m k_i = n$ and depth m in terms of which one can express all polyzeta values of same weight and depth using rational coefficients. These generating polyzeta values are called irreducible. On basis of a numerical work they ended up to several conjectures, one of them being that the number M_n of irreducible polyzeta values of weight n is coded by the generating function $1 - x^2 - x^3 = \prod_n (1 - x^n)^{M_n}$. They also demonstrated that for $3 \leq n \leq 9$ M_n enumerates so called positive knots with n crossings. The results emerged from the study of divergences of Feynman diagrams of ϕ^4 theory. A rule for assigning knots to Feynman diagrams allows to predict the level of transcendentality characterized by M_n associated with the counter term coefficients of a given UV divergent Feynman diagram.

p-Adicization philosophy inspires the following conjecture about the values of polyzetas. From $\zeta(2) = \pi^2/6$, and $\zeta(4) = \pi^4/90$, and more general result $\zeta(2n) \propto \pi^{2n}$, one could guess that that the values of polyzetas at the level $K = n$ are proportional to $x\pi^n$, where x is a number which belongs to an extension of rationals defining a finite extension of p-adic numbers (say combination of algebraic number and root of e). Hence the identities between polyzetas would make sense for finite extensions of p-adic numbers. Also Drinfeld's associator would make sense for these extensions.

3. Do the zeros of polyzetas correspond to the values of off mass shell super-canonical conformal weights?

The following considerations lend support to the speculation that Riemann Zeta and polyzetas are deeply related to the basic structure of quantum TGD so that Riemann Hypothesis might have much deeper physical content than previously thought.

1. The complex conformal weights associated with the super-canonical algebra could correspond to the zeros of Riemann Zeta or of their subset.

2. The intermediate particles appearing in stringy tree diagrams could have off mass shell super canonical conformal weights. This would make possible braiding during off mass-shell propagation and the interpretation as a time evolution for a braid would make sense. The zeros of polyzetas forming $2(n-1)$-dimensional surfaces might correspond to conformal weights of intermediate n-particle states appearing in the generalized Feynman diagrams involving n incoming particles in interaction. Also conformal weights of particles forming bound states could satisfy this condition, and one generalize the notion of bound state energy to that of "bound state conformal weight". According to the proposal of [E10], the braiding of join along boundaries bonds is the space-time correlate for bound state entanglement. At the level of configuration the braiding in the space of conformal weights would be in question.

   (a) The symmetrized 2-zeta $P_2(a_1, a_2)$, which might be relevant at least in the case of identical bosons resulting in the decay $a \rightarrow b+c$ or its reversal. The pairs of integers (m, n), $m+n = -2k$, $k > 0$ are its trivial zeros as one might expect from the addition of conformal weights. Since symmetrized n-zeta is expressible in terms of lower symmetrized poly-zetas the same holds true for arbitrary number of particles.

   (b) In 2-particle intermediate (or bound) state the conformal weights of interacting particles must belong to a 2-dimensional surface of non-trivial zeros of $P_2(a_1, a_2)$. Let $z_i = 1/2 + y_i$, $i = 1, 2$ be two non-trivial zeros of ζ. The pairs (a_1, a_2), where $a_1 = 1/2 + iy_1$, $a_2 = z_2 - z_1$ so that $a_1 + a_2$ is zero of ζ, define nontrivial zeros of $P_2(a_1, a_2)$. A possible interpretation for a_2 would be as a conformal weight of an off mass shell virtual particle resulting when the intermediate state decays to an on mass shell particle and virtual particle. The on mass shell decay of the intermediate state $a \to b + c$ is forbidden by Riemann hypothesis unless one allows trivial zeros as conformal weights. The conservation of imaginary part of conformal weight for $a+b \to c+d$ by exchange in t-channel yields $(y_a, y_c) = (y_b, y_d)$.

(c) For the symmetrized 3-zeta $P_3(a_1, a_2, a_3)$ the zeros define a 4-dimensional surface. The study of the expression of P_3 given above shows that in this case there are no zeros expressible as combinations of non-trivial zeros of ζ. The emission of a_1 as on-mass shell particle from the 3-particle intermediate state would imply that a_1 and $a_1 + a_2 + a_3$ correspond to zeros of ζ. $a_2 + a_3$ would be a difference for zeros of ζ. The remaining condition would be $\zeta(a_2)P_2(a_1, a_3) + \zeta(a_3)P_2(a_1, a_2) = 0$, which allows only discrete points as solutions for given values of a_1 and $a_2 + a_3$. $a_2 + a_3$ has interpretation as an intermediate state and could decay to two on mass shell particles.

4. Does quantum criticality imply the identification of conformal weights as zeros of polyzetas?

The physical justification for the identification of zeros of polyzetas as allowed super-canonical conformal weights comes from quantum criticality.

1. Riemann Zeta has interpretation as a partition function [E8]. Also polyzetas might have a similar interpretation. Partition functions vanish at critical points representing phase transitions so that the vanishing of Riemann Zeta for allowed free particle conformal weights is consistent with quantum criticality, and allowed conformal weights would be analogous to critical temperature.

2. Quantum criticality means the vanishing of loop corrections. The brave guess is that for given values of super-canonical conformal weights the loop corrections predicted by TGD are proportional to polyzeta values just as they are in ordinary quantum field theories. The difference would be that conformal weights can be complex in the case of super-canonical algebra. The vanishing of loop corrections at criticality would force the identification of the super-canonical conformal weights as zeros of polyzetas.

3. The integer valued arguments of polyzetas correlate strongly with the number of loops correlating in turn with the superficial divergence of

the Feynman diagram. A multi-loop specialist could probably immediately tell whether also in quantum field context the integers appearing as arguments of polyzeta could have an interpretation as conformal weights. In [22] a simple quantum field theoretical model yielding the values of polyzetas as vacuum diagrams is constructed, and vacuum diagrams are given by $\lambda^n \times (g\overline{g})^{2m+1}\zeta(k_1, ..., k_m)$. λ has dimension of inverse length so that the weight $n = \sum_i k_i$ indeed appears as a scaling dimension of the vacuum diagram. The depth m defines the powers of the coupling constants g and $\overline{g}$ associated with the vacuum diagram. Unfortunately this model is purely formal so that one cannot draw strong conclusions from it.

4 Hopf algebras and ribbon categories as basic structures

In this section the basic notions related to Hopf algebras and categories are discussed from TGD point of view. Examples are left to appendix. The new element is the graphical representation of the axioms leading to the idea about the equivalent of loop diagrams and tree diagrams based on general algebraic axioms.

4.1 Hopf algebras and ribbon categories very briefly

An algebraic formulation generalizing braided Hopf algebras and related structures to what might be called quantum category would involve the replacement of the co-product of Hopf algebras with morphism of quantum category having as its objects the Clifford algebras associated with configuration space spinor structure for various 3-topologies. The corresponding Fock spaces would would define algebra modules and the objects of the category would consists of pairs of algebras and corresponding modules. The underlying primary structure would be second quantized free induced spinor fields associated with 3-surfaces with various 3-topologies and generalized conformal structures.

1. Bi-algebras

Bi-algebras have two algebraic operations. Besides ordinary multiplication $\mu : H \otimes H \rightarrow H$ there is also co-multiplication $\Delta : H \rightarrow H \otimes H$. Algebra satisfies the associativity axiom (Ass): $a(bc) = (ab)c$, or more formally, $\mu(id \otimes \mu) = \mu(\mu \otimes id)$, and the unit axiom (Un) stating that there is morphism $\eta : k \rightarrow A$ mapping the unit of A to the unit of field k. Commutativity axiom (Co) $ab = ba$ translates to $\mu \otimes \tau \equiv \mu^{op} = \mu$, where τ permutes factors in tensor product $A \otimes A$.

Δ satisfies mirror images of these axioms. Co-associativity axiom (Coass) reads as $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$, co-unit axiom (Coun) states existence of morphism $\epsilon : k \rightarrow C$ mapping the unit of A to that of k, and co-commutativity (Coco) reads as $\tau \circ \Delta \equiv \Delta^{op} = \Delta$. For a bi-algebra H also additional axioms are satisfied: in particular, Δ (μ) acts as algebra (bi-algebra) morphism. When represented graphically, this constraint states that a box diagram is equivalent to a tree diagram as will be found and served as the stimulus for the idea that loop diagrams might be equivalent with tree diagrams.

Left and right algebra modules and algebra representations are defined in an obvious manner and satisfy associativity and unit axioms. A left co-module corresponds a pair (V, Δ_V) where the co-action $\Delta_N : V \rightarrow A \otimes V$ satisfies co-associativity and co-unit axioms. Right co-module is defined in an analogous manner.

Particle fusion $A \otimes B \rightarrow C$ corresponds to μ: $A \otimes B \rightarrow C = AB$. Co-multiplication Δ corresponds time reversal $C \rightarrow A \otimes B$ of this process, which is kind of a time-reversal for multiplication. The generalization would mean that μ and Δ become morphisms $\mu : B \otimes C \rightarrow A$ and $\Delta : A \rightarrow B \otimes C$, where A, B, C are objects of the quantum category. They could be either representations of same algebra or even different algebras.

2. Drinfeld's quantum double

Drinfeld's quantum double [18, 19] is a braided Hopf algebra obtained by combining Hopf algebra $(H, \mu, \Delta, \eta, \epsilon, S, R)$ and its dual $H^\star$ to a larger Hopf algebra known as quasi-triangular Hopf algebra satisfying $\Delta = R\Delta^{op}R^{-1}$, where $\Delta^{op}(a)$ is obtained by permuting the two tensor factors. Duality means existence of a scalar product and the two algebras correspond to Hermitian conjugates of each other.

In TGD framework the physical states associated with these algebras have opposite energies since in TGD framework antimatter (or matter depending on the phase of matter) corresponds to negative energy states. The states of the Universe would correspond to states with vanishing conserved quantum numbers, and in concordance with crossing symmetry, particle reactions could be interpreted as transitions generating zero energy states from vacuum.

The notion of duality [18] is needed to define an inner product and S-matrix. Essentially Dirac's bra-ket formalism is in question. The so called evaluation map $ev : V \otimes V^{\star} \rightarrow k$ defined as $ev(v^i \otimes v_j) = \langle v^i, v_j \rangle = \delta_{ij}$ defines an inner product in any Hopf algebra module. The inverse of this map is the linear map $k \rightarrow V$ defined by $\delta_v(1) = v_i \otimes v^i$. For a tensor category with unit I, field k is replaced with unit I, and left duality these maps are replaced with maps $b_V : I \rightarrow V \otimes V^{\star}$ and $d_V = V \otimes V^{\star} \rightarrow I$. Right duality is defined in an analogous manner. The map d_V assigns to a given zero energy state S-matrix element. Algebra morphism property $b_V(ab) = b_V(a)b_V(b)$ would mean that the outcome is essentially the counterpart of free field theory Feynman diagram. This diagram is convoluted with the S-matrix element coded to the entanglement coefficients between positive and negative energy particles of zero energy state.

3. Ribbon algebras and ribbon categories

The so called ribbon algebra [18] is obtained by replacing one-dimensional strands with ribbons and adding to the algebra the so called twist operation θ acting as a morphism in algebra and in any algebra module. Twist allows to introduce the notion of trace, in particular quantum trace.

The thickening of one-dimensional strands to 2-dimensional ribbons is especially natural in TGD framework, and corresponds to a replacement of points of time=constant section of 4-surface with one-dimensional curves along which the S-matrix defined by R-matrix is constant. Ribbon category is defined in an obvious manner. There is also a more general definition of ribbon category with objects identified as representations of a given algebra and allowing morphisms with arbitrary number of incoming and outgoing strands having interpretation as many-particle vertices in TGD framework. The notion of quantum category defined as a generalization of a ribbon category involving the generalization of algebra product and co-product as morphisms between different objects of the category and allowing objects to correspond different algebras might catch the essentials of the physics of TGD Universe.

4.2 Algebras, co-algebras, bi-algebras, and related structures

It is useful to formulate the notions of algebra, co-algebra, bi-algebra, and Hopf algebra in order to understand how they might help in attempt to formulate more precisely the view about what generalized Feynman diagrams could mean. Since I am a novice in the field of quantum groups, the definitions to be represented are more or less as such from the book "Quantum Groups" of Christian Kassel [18] with some material (such as the construction of Drinfeld double) taken from [19]. What is new is a graphical representation of algebra axioms and the proposal that algebra and co-algebra operations have interpretation in terms of generalized Feynman diagrams.

In the following considerations the notation id_k for the isomorphism $k \to k \otimes k$ defined by $x \to x \otimes x$ and its inverse will be used.

4.2.1 Algebras

Algebra can be defined as a triple (A, μ, η), where A is a vector space over field k and $\mu : A \otimes A \to A$ and $\eta : k \to A$ are linear maps satisfying the following axioms (Ass) and (Un).

(Ass): The square

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes id} & A \otimes A \\ \downarrow id \otimes \mu & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array} \tag{8}$$

commutes.

(Un): The diagram

$$\begin{array}{ccccc} k \otimes A & \xrightarrow{\eta \otimes id} & A \otimes A & \xleftarrow{id \otimes \eta} & A \otimes k \\ & \searrow^{\cong} & \downarrow \mu & \swarrow_{\cong} & \\ & & A & & \end{array} \tag{9}$$

commutes. Note that η imbeds field k to A.

(Comm) If algebra is commutative, the triangle

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau_{A,A}} & A \otimes A \\ & \searrow^{\mu} \quad \swarrow_{\mu} & \\ & A & \end{array} \tag{10}$$

commutes. Here $\tau_{A,A}$ is the flip switching the factors: $\tau_{A,A}(a \otimes a') = a' \otimes a$.

A morphism of algebras $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is a linear map $A \to A'$ such that

$$\mu' \circ (f \otimes f) = f \circ \mu, \quad \text{and} \quad f \circ \eta = \eta' \ .$$

A graphical representation of the algebra axioms is obtained by assigning to the field k a dashed line to be referred as a vacuum line in the sequel and to A a full line, to η a vertex $\times$ at which k-line changes to A-line. The product μ can be represented as 3-particle vertex in which algebra lines fuse together. The three axioms (Ass), (Un) and (Comm) can are expressed graphically in figure 4.2.1.

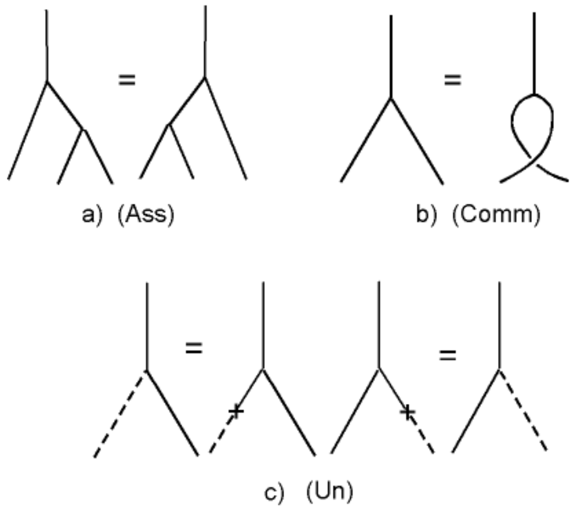


Figure 1: Graphical representation for the axioms of algebra. a) $a(bc) = (ab)c$, b) $ab = ba$, c) $ka = \mu(\eta(k), a)$ and $ak = \mu(a, \eta(k))$.

Note that associativity axiom implies that two tree diagrams not equivalent as Feynman diagrams are equivalent in the algebraic sense.

4.2.2 Co-algebras

The definition of co-algebra is obtained by systematically reversing the directions of arrows in the previous diagrams.

A co-algebra is a triple(C, Δ, ϵ), where C is a vector space over field k and $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ are linear maps satisfying the following axioms (Coass) and (Coun).

(Coass): The square

$$\begin{array}{ccc} C & \stackrel{\Delta}{\longrightarrow} & C \otimes C \\ \downarrow\Delta & & \downarrow id \otimes \Delta \\ C \otimes C & \stackrel{\Delta \otimes id}{\longrightarrow} & C \otimes C \otimes C \end{array} \tag{11}$$

commutes.

(Coun): The diagram

$$\begin{array}{ccccc} k\otimes C & \xleftarrow[\epsilon\otimes id]{} & C\otimes C & \xrightarrow{id\otimes\epsilon} & C\otimes k \\ & \nwarrow{\scriptstyle\cong} & \Delta\uparrow & \nearrow{\scriptstyle\cong} & \\ & & C & & \end{array} \qquad (12)$$

commutes. The map Δ is called co-product or co-multiplication whereas ϵ is called the counit. The commutative diagram state that the co-product is co-associative and that co-unit commutes with co-product.

(Cocomm) If co-algebra is commutative, the triangle

$$\begin{array}{ccc} & C & \\ \swarrow{\scriptstyle\Delta} & & \searrow{\scriptstyle\Delta} \\ C\otimes C & \xrightarrow{\tau_{C,C}} & C\otimes C \end{array} \qquad (13)$$

commutes. Here $\tau_{C,C}$ is the flip switching the factors: $\tau_{C,C}(c\otimes c') = c'\otimes c$.

A morphism of co-algebras $f : (C,\Delta,\epsilon) \to (C',\Delta',\epsilon')$ is a linear map $C \to C'$ such that

$$(f\otimes f)\circ\Delta = \Delta'\circ f \ \ , \ \text{and} \ \ \epsilon = \epsilon'\circ f \ \ .$$

It is straightforward to define notions like co-ideal and co-factor algebra by starting from the notions of ideal and factor algebra. A very useful notation is Sweedler's sigma notation for $\Delta(x)$, $x\in C$ as element of $C\otimes C$:

$$\Delta(x) = \sum_i x_i'\otimes x_i'' \equiv \sum_{\{x\}} x'\otimes x'' \ \ .$$

Also co-algebra axioms allow graphical representation. One assigns to ϵ a vertex $\times$ at which C-line changes to k-line: the interpretation is as an absorption of a particle by vacuum. The co-product Δ can be represented as 3-particle vertex in which C-line decays to two C-lines. The graphical representation of the three axioms (Coass), (Coun), and (Cocomm) is related to the representation of algebra axioms by "time reversal", that is turning the diagrams for the algebra axioms upside down (see figure 4.2.2).

4.2.3 Bi-algebras

Consider next a vector space H equipped simultaneously with an algebra structure (H,μ,η) and a co-algebra structure (H,Δ,ϵ). There are some

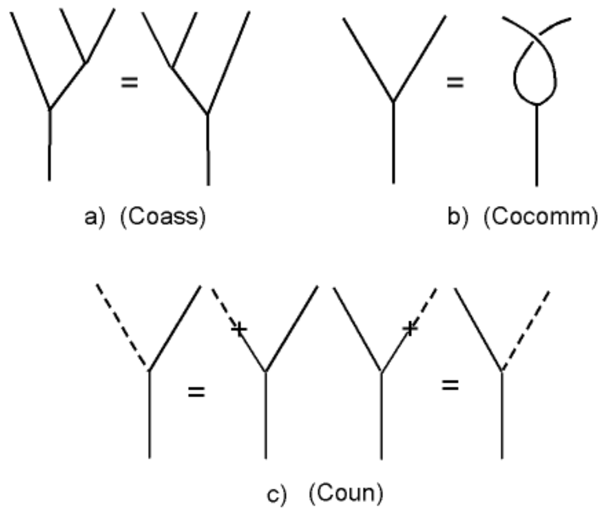


Figure 2: Graphical representation for the axioms of co-algebra is obtained by turning the representation for algebra axioms upside down. a) $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$, b) $\Delta = \Delta^{op}$, c) $(\epsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id$.

compatibility conditions between these two structures. $H \otimes H$ can be given the induced structures of a tensor product of algebras and of co-algebras.

The following two statements are equivalent.

1. The maps μ and η are morphisms of co-algebras. For μ this means that the diagrams

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\mu} & H \\ \Big\downarrow (id \otimes \tau \otimes id) \otimes (\Delta \otimes \Delta) & & \Big\downarrow \Delta \\ (H \otimes H) \otimes (H \otimes H) & \xrightarrow{\mu \otimes \mu} & H \otimes H \end{array} \tag{14}$$

and

$$
\begin{array}{ccc}
H \otimes H & \xrightarrow{\epsilon \otimes \epsilon} & k \otimes k \\
\Big\downarrow \mu & & \Big\downarrow id \\
H & \xrightarrow{\epsilon} & k
\end{array}
\tag{15}
$$

commute. For η this means that the diagrams

$$
\begin{array}{ccc}
k & \xrightarrow{\eta} & H \\
\Big\downarrow id & & \Big\downarrow \Delta \\
k \otimes k & \xrightarrow{\eta \otimes \eta} & H \otimes H
\end{array}
\qquad
\begin{array}{ccc}
k & \xrightarrow{\eta} & H \\
& {\scriptstyle id}\searrow \quad \swarrow{\scriptstyle \epsilon} & \\
& k &
\end{array}
\tag{16}
$$

commute.

2. The maps Δ and ϵ are morphisms of algebras.
   For Δ this means that diagrams

$$
\begin{array}{ccc}
H \otimes H & \xrightarrow{\Delta \otimes \Delta} & (H \otimes H) \otimes (H \otimes H) \\
\Big\downarrow \mu & & \Big\downarrow (\mu \otimes \mu)(id \otimes \tau \otimes id) \\
H & \xrightarrow{\Delta} & H \otimes H
\end{array}
\tag{17}
$$

and

$$
\begin{array}{ccc}
k & \xrightarrow{\eta} & H \\
\Big\downarrow id & & \Big\downarrow \Delta \\
k \otimes k & \xrightarrow{\eta \otimes \eta} & H \otimes H
\end{array}
\tag{18}
$$

commute.

For ϵ this means that the diagrams

$$\begin{array}{ccc} H\otimes H & \xrightarrow{\epsilon\otimes\epsilon} & k\otimes k \\ \downarrow \mu & & \downarrow id \\ H & \xrightarrow{\epsilon} & k \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{\eta} & H \\ & \searrow^{id} \quad \swarrow_{\epsilon} & \\ & k & \end{array} \tag{19}$$

commute. The proof of the theorem involves the comparison of the commutative diagrams expressing both statements to see that they are equivalent.

The theorem inspires the following definition.

Definition: A bi-algebra is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$, where (H, μ, η) is an algebra and (H, Δ, ϵ) is co-algebra satisfying the mutually equivalent conditions of the previous theorem. A morphisms of bi-algebras is a morphism for the underlying algebra and bi-algebra structures.

An element $x \in H$ is known as primitive if one has $\Delta(x) = 1 \otimes x + x \otimes 1$ and have $\epsilon(x) = 0$. The subspace of primitive elements is closed with respect to the commutator $[x, y] = xy - yx$. Note that for primitive elements $\mu \circ \Delta = 2id_H$ holds true so that $\mu/2$ acts as the left inverse of Δ.

Given a vector space V, there exists a unique bi-algebra structure on the tensor algebra $T(V)$ such that $\Delta(v) = 1 \otimes v + v \otimes 1$ and $\epsilon(v) = 0$ for any element v of V. By the symmetry of Δ this bi-algebra structure is co-commutative and corresponds to the "classical limit". Also the Grassmann algebra associated with V allows bi-algebra structure defined in the same manner.

Figure 4.2.3 provides a representation for the axioms of bi-algebra stating that Δ and ϵ act as algebra morphisms of algebra and or equivalent that μ and η act as co-algebra morphisms. The axiom stating that Δ (μ) is algebra (co-algebra) morphism implies that scattering diagrams differing by a box loop are equivalent. The statement that μ is co-algebra morphism reads $(id\otimes\mu\otimes id)(\Delta\otimes\Delta) = \Delta\circ\mu$ whereas the mirror statement $\Delta(ab) = \Delta(a)\Delta(b)$ for Δ reads as $\Delta \circ \mu = \mu(\Delta \otimes \Delta)$ and gives rise to the same graph.

4.2.4 Hopf algebras

Given an algebra (A, μ, η) and co-algebra (C, Δ, ϵ), one can define a bilinear map, the convolution on the vector space $Hom(C, A)$ of linear maps from C to A. By definition, if f and g are such linear maps, then the convolution $f \star g$ is the composition of the maps

$$C \underset{\Delta}{\rightarrow} C\otimes C \underset{f\otimes g}{\rightarrow} A\otimes A \underset{\mu}{\rightarrow} A \tag{20}$$

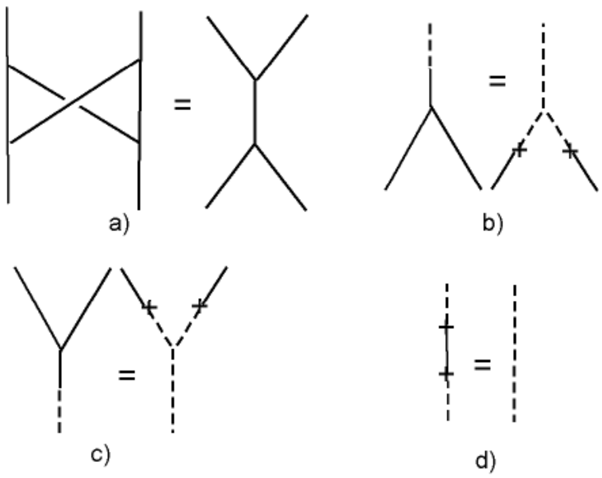


Figure 3: Graphical representation for the conditions guaranteing that μ and η (Δ and ϵ) act as homomorphisms of co-algebra (algebra). a)$(id \otimes \mu \otimes id)(\Delta \otimes \Delta) = \Delta \circ \mu$, b) $\epsilon \circ \mu = id \circ (\epsilon \otimes \epsilon)$, c) $\Delta \circ \eta = \mu \otimes \circ id_k$, d) $\epsilon \circ \eta = id_k$.

Using Sweedler's sigma notion one has

$$f \star g(x) = \sum_{\{x\}} f(x')g(x'') \ . \tag{21}$$

It can be shown that the triple $(Hom(C,A), \star, \Delta, \eta \circ \epsilon)$ is an algebra and that the map $\Lambda_{C,A} : A \otimes C^\star \rightarrow Hom(C,A)$ defined as

$$\Lambda_{C,A}(a \otimes \gamma)(c) = \gamma(c)a$$

is a morphism of algebras, where $C^\star$ is the dual of the finite-dimensional co-algebra C.

For $A = C$ the result gives a mathematical justification for the crossing symmetry inspired re-interpretation of the unitary S-matrix interpreted usually as an element of $Hom(A,A)$ as a state generated by element of $A \otimes A^\star$ from the vacuum $|vac\rangle = |vac_A\rangle \otimes |vac_{A^\star}\rangle$. This corresponds to the interpretation of the reaction $a_i|vac_A\rangle \rightarrow a_f|vac_A\rangle$ as a transition creating state $a_i \otimes a_f^\star|vac\rangle$ with vanishing conserved quantum numbers from vacuum.

With these prerequisites one can introduce the notion of Hopf algebra.

Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bi-algebra. An endomorphism S of H is called an antipode for the bi-algebra H if

$$S \star id_H = id_H \star S = \eta \circ \epsilon \ .$$

A Hopf algebra is a bi-algebra with an antipode. A morphism of a Hopf algebra is a morphism between the underlying bi-algebras commuting with the antipodes.

The graphical representation of the antipode axiom is given in the figure below.

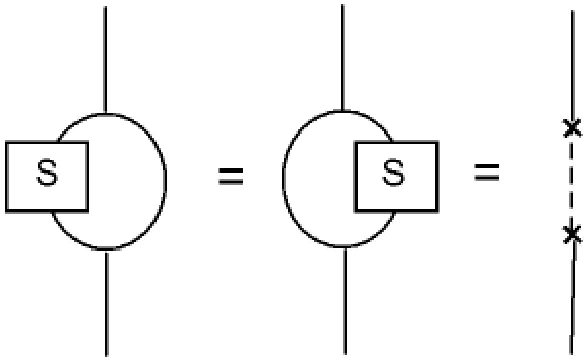


Figure 4: Graphical representation of antipode axiom $S \star id_H = id_H \star S = \eta \circ \epsilon$.

The notion of scalar product central for physical applications boils down to the notion of duality. Duality between Hopf algebras U and H means the existence of a morphism $x \to \Psi(x)$: $H \to U^\star$ defined by a bilinear form $\langle u, x\rangle = \Psi(x)(u)$ on $U \times H$, which is a bi-algebra morphism. This means that the conditions

$$\begin{aligned} &\langle uv, x\rangle = \langle u \otimes v, \Delta(x)\rangle \ , \quad &&\langle u, xy\rangle = \langle \Delta(u), x \otimes y\rangle \ , \\ &\langle 1, x\rangle = \epsilon(x) \ , &&\langle u, 1\rangle = \epsilon(u) \ , \\ &\langle S(u), x\rangle = \langle u, S(x)\rangle \end{aligned} \tag{22}$$

are satisfied. The first condition on multiplication and co-multiplication, when expressed graphically, states that the decay $x \to u \otimes v$ can be regarded as time reversal for the fusion of $u \otimes v \to x$. Second condition has analogous interpretation.

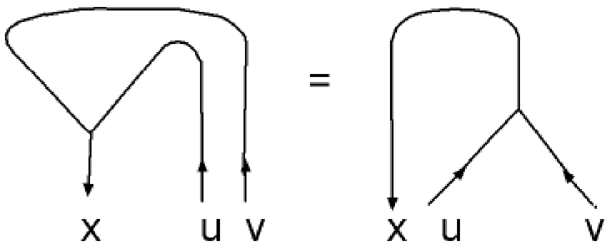


Figure 5: Graphical representation of the duality condition $\langle uv, x\rangle = \langle u \otimes v, \Delta(x)\rangle$.

4.2.5 Modules and comodules

Left and right algebra modules and algebra representations are defined in an obvious manner and satisfy associativity and unit axioms having diagrammatic representation similar to that for corresponding algebra axioms.

A left co-module corresponds a pair (V, Δ_V), where the co-action Δ_N: $V \to C \otimes V$ satisfies co-associativity axiom $(id_C \otimes \Delta_N) \circ \Delta_N = (\Delta \otimes id_N) \circ \Delta_N$ and co-unit axiom $(\epsilon \otimes id) \circ \Delta_N = id_N$. A right co-module is defined in an analogous manner. It is convenient to introduce Sweedlers's notation for Δ_N as $\Delta_N = \sum_{\{c\}} x_C \otimes x_N$.

One can define module and comodule morphisms and tensor product of modules and co-modules in a rather obvious manner. The module N could be also algebra, call it A, in which case μ_A and η_A are assumed to act as H-comodule morphisms.

The standard example is quantum plane $A = M(2)_q$ is the free algebra generated variables x, y subject to to relations $yx = qxy$ and having coefficients in k. The action of Δ_A reads as

$$\Delta_A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} \ .$$

Δ_A defines algebra morphism from A to $SL(2)_q \otimes A$: $\Delta_a(yx) = \Delta_A(y)\Delta_A(x) = q\Delta_A(x)\Delta_A(y) = \Delta(qxy)$.

4.2.6 Braided bi-algebras

$\Delta^{op} = \tau_{H,H} \circ \Delta$ defines the opposite co-algebra H^{op} of H. A braided bi-algebra $(H, \mu, \eta, \Delta, \epsilon)$ is called quasi-co-commutative (or quasi-triangular) if there exists an element R of algebra $H \otimes H$ such that for all $x \in H$ one has

$$\Delta^{op} = R \Delta R^{-1} \quad .$$

One can express R in the form

$$R = \sum_i s_i \otimes t_i \quad .$$

It is convenient to denote by R_{ij} the R matrix acting in i^{th} and j^{th} tensor factors of n^{th} tensor power of H. More precisely, R_{ij} can be defined as an operator acting in an n-fold tensor power of H by the formula $R_{ij} = y^{(1)} \otimes y^{(2)} \otimes ... \otimes y^{(p)}$, $p \leq n$, $y^{(k_i)} = s_i$ and $y^{(k_j)} = t_j$, $y^{(k)} = 1$ otherwise. For instance, one has $R_{13} = \sum_i s_i \otimes 1 \otimes t_i$.

With these prerequisites one can define a braided bi-algebra as a quasi-commutative bi-algebra $(H, \mu, \eta, \Delta, \epsilon, S, S^{-1}, R)$ as an algebra with a preferred element $R \in H \otimes H$ satisfying the two relations

$$\begin{aligned} (\Delta \otimes id_H)(R) &= R_{13} R_{23} \quad , \\ (id_H \otimes \Delta)(R) &= R_{13} R_{12} \quad . \end{aligned} \tag{23}$$

Braided bi-algebras, known also as quasi-triangular bi-algebras, are central in the theory of quantum groups, R-matrices, and braid groups. By a direct calculations one can verify the following relations.

1. Yang-Baxter equations

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad , \tag{24}$$

and the relation

$$(\epsilon \otimes id_H)(R) = 1 \tag{25}$$

hold true.

2. Since H has an invertible antipode S, one has

$$\begin{aligned}(S \otimes id_H)(R) &= R^{-1} = (id_H \otimes S^{-1})(R) \ , \\ (S \otimes S)(R) &= R \ . \end{aligned} \tag{26}$$

The graphical representation of the Yang-Baxter equation in terms of the relations of braid group generators is given in the figure 4.2.6.

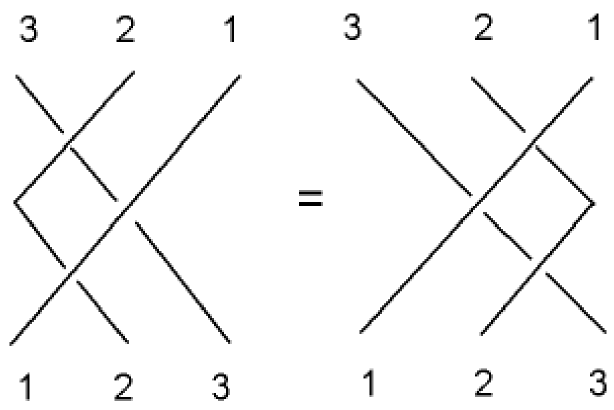


Figure 6: Graphical representation of Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

4.2.7 Ribbon algebras

Let H be a braided Hopf algebra with a universal matrix $R = \sum_i s_i \otimes t_i$ and set $u = \sum_i S(t_i)s_i$. It can be shown that u is invertible with the inverse $u^{-1} = \sum s_i S^2(t_i)$ and that $uS(u) = S(u)u$ is central element in H. Furthermore, one has $\epsilon(u) = 1$ and $\Delta(u) = (R_{21}R)^{-1}(u \otimes u)$, and the antipode is given for any $x \in H$ by $S^2(x) = uxu^{-1}$.

Ribbon algebra has besides $R \in H \otimes H$ also a second preferred element called θ. A braided Hopf algebra is called ribbon algebra if there exists a central element θ of H satisfying the relations

$$\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta) \ , \quad \epsilon(\theta) = 1 \ , \quad S(\theta) = \theta \ . \tag{27}$$

It can be shown that θ^2 acts like $S(u)u$ on any finite-dimensional module [18].

4.2.8 Drinfeld's quantum double

Drinfeld's quantum double construction allows to build a quasi-triangular Hopf algebra by starting from any Hopf algebra H and its dual $H^\star$, which exists in a finite-dimensional case always, and as a vector space is isomorphic with H. Besides duality normal ordering is second ingredient of the construction. Physically the generators of the algebra and its dual correspond to creation and annihilation operator type operators. Drinfeld's quantum double construction is represented in a very general manner in [18]. A construction easier to understand by a physicist is discussed in [19]. For this reason this representation is summarized here although the style differs from the representation of [18] followed in the other parts of appendices.

Consider first what is known.

1. Duality means the existence of basis $\{e_a\}$ for H and $\{e^a\}$ for $H^\star$ and inner product (or evaluation as it is called in [18]) $ev : H^\star \otimes H \to k$ defined as $ev(e^a e_b) \equiv \langle e^a, e_b \rangle = \delta^a_b$ and its inverse $\delta : k \to H^\star \otimes H$ defined by $\delta(1) = e^a e_a$. One can extend the inner product to an inner product in the tensor product $(H^\star \otimes H^\star) \otimes (H \otimes H)$ in an obvious manner.

2. The product (co-product) in H $(H^\star)$ coincides with the co-product (product) in $H^\star$ (H) in the sense that one has

$$\begin{aligned} \langle e^c, e_a e_b \rangle = m^c_{ab} = \langle \Delta(e^c), e_b \otimes e_a \rangle \ , \\ \langle e^a e^b, e_c \rangle = \mu^{ab}_c = \langle e^a \otimes e^b, \Delta(e_c) \rangle \ , \end{aligned} \tag{28}$$

   These equations are quite general expressions for the duality expressed graphically in figure 4.2.4.

3. The antipodes S for H and $H^\star$ can be represented as matrices

$$S_H(e_a) = S_a^{\ b} e_b \ , \quad S_{H^\star}(e^a) = (S^{-1})^a_{\ b} e^b \ . \tag{29}$$

The task is to construct algebra product μ and co-algebra product Δ, unit η and co-unit ϵ, antipode, and R-matrix R for for $H \otimes H^\star$. The natural basis for $H \otimes H^\star$ consists of $e_a \otimes e^b$.

1. Co-product Δ is simply the product of co-products

$$\Delta(e_a e^b) = \Delta(e_a)\Delta(e^b) = m^b_{vu}\mu^{cd}_a e_c e^u \otimes e_d e^v \ . \tag{30}$$

2. Product μ involves normal ordering prescription allowing to transform products $e^a e_b$ (elements of $H^\star \otimes H$) to combinations of basis elements $e_a e^b$ (elements of $H \otimes H^\star$. This map must be consistent with the requirement that co-product acts as an algebra morphism. Drinfeld's normal ordering prescription, or rather a map $c_{H^\star,H}$: $H^\star \otimes H \rightarrow H \otimes H^\star$ is given by

$$c_{H^\star,H}(e^a e_b) = R^{ac}_{bd} e_c e^d \ , \quad R^{ac}_{bd} = m^x_{kd} m^a_{xu} \mu^{vy}_b \mu^{ck}_y (S^{-1})^u_v e_c e^d \ . \tag{31}$$

   The details of the formula are far from being obvious: the axioms of tensor category with duality to be discussed later might allow to relate $R_{H^\star,H}$ to $R_{H,H}$ and this might help to understand the origin of the expression. Normal ordering map can be interpreted as braid operation exchanging H and $H^\star$ and the matrix defining the map could be regarded as R-matrix $R_{H\otimes H^\star}$.

3. The universal R-matrix is given by

$$R \ = \ (e_a \otimes id_{H^\star}) \otimes (id_H \otimes e^a) \ , \tag{32}$$

   where the summation convention is applied. One can show that $R\Delta = \Delta^{op} R$ by a direct calculation.

4. The antipode $S_{H\otimes H^\star}$ follows from the product of antipodes for H and $H^\star$ using the fact that antipode is antihomomorphism using the normal ordering prescription

$$S_{H\otimes H^\star}(e_a e^b) = c_{H^\star,H}(S(e^b)S(e_a)) \ . \tag{33}$$

4.2.9 Quasi-Hopf algebras and Drinfeld associator

Braided Hopf algebras are quasi-commutative in the sense that one has $\Delta^{op} = R\Delta R^{-1}$. Also the strict co-associativity can be given up and this means that one has

$$(\Delta \otimes id)\Delta = \Phi(id \otimes \Delta)\Phi^{-1} \quad , \tag{34}$$

where $\Phi \in H \otimes H \otimes H$ is known as Drinfeld's associator and appears in the of conformal fields theories. If the resulting structure satisfies also the so called Pentagon Axiom (to be discussed later, see Eq. 45 and figure 4.3.2), it is called quasi-Hopf algebra. Pentagon Axiom boils down to the condition

$$(id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi) = (id \otimes \Phi)(id \otimes id \otimes \Delta)(\Phi)(\Phi \otimes id) \quad . \tag{35}$$

The Yang-Baxter equation for quasi-Hopf algebra reads as

$$R_{12}\Phi_{312}R_{13}\Phi^{-1}_{1322}R_{23}\Phi_{123} = \Phi_{321}R_{23}\Phi^{-1}_{231}R_{13}\Phi_{213}R_{12}\Phi_{123} \quad . \tag{36}$$

The left-hand side arises from a sequence of transformations

$$(12)3 \xrightarrow{\Phi_{123}} 1(23) \xrightarrow{R_{23}} 1(32) \xrightarrow{\Phi^{-1}_{132}} (31)2 \xrightarrow{R_{13}} 3(12) \xrightarrow{\Phi_{312}} 3(12) \xrightarrow{R_{12}} 3(21) \quad . \tag{37}$$

The right-hand side arises from the sequence

$$(12)3 \xrightarrow{R_{12}} (21)3 \xrightarrow{\Phi_{213}} 2(13) \xrightarrow{R_{13}} 2(31) \xrightarrow{\Phi^{-1}_{231}} (23)1 \xrightarrow{R_{23}} (32)1 \xrightarrow{\Phi_{321}} 3(21) \quad . \tag{38}$$

One can produce new quasi-Hopf algebras by gauge (or twist) transformations using invertible element $\Omega \in H \otimes H$ called twist operator

$$\begin{aligned} \Delta(a) &\rightarrow \Omega\Delta(a)\Omega^{-1} \quad , \\ \Phi &\rightarrow \Omega_{23}(id \otimes \Delta)(\Omega)\Phi(\Delta \otimes id)(\Omega^{-1})\Omega^{-1}_{12} \quad , \\ R &\rightarrow \Omega R\Omega^{-1} \quad . \end{aligned} \tag{39}$$

Quasi-Hopf algebras appear in conformal field theories and correspond quantum universal enveloping algebras divided by their centralizer. Consider as an example the R-matrix R^{j_1,j_2} relating $j_1 \otimes j_2$ and $j_2 \otimes j_1$ representations $\Delta^{j_1,j_2}(a)$ and $\Delta^{j_2,j_1}(a)$ of the co-product Δ of $U(sl(2))_q$. $\Delta^{j,j}(a)$ commutes with R^{jj} for all elements of the quantum group. The action of $g_i = qR^{jj}$ acting in i^{th} and $(i+1)^{th}$ tensor factors extends to the representation $(V_j)^{\times n}$ in an obvious manner. From the Yang-Baxter equation it follows that the operators g_i define a representation of braid group B_n:

$$\begin{aligned} g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \ , \\ g_i g_j &= g_j g_i \ , \ \text{for } |j-k| \geq 2 \ . \end{aligned} \quad (40)$$

Under certain conditions the braid group generators generate the whole centralizer C_q^n for the representation of quantum group. For instance, this occurs for $j = 1/2$. In this case the additional condition

$$g_i^2 = (q^2-1)g_i + q^2 \times 1 \ , \quad (41)$$

so that the centralizer is isomorphic with the Hecke algebra $H_n(q)$, which can be regarded as a q-deformation of permutation group S_n.

The result generalizes. In Wess-Zumino-Witten model based on group G the relevant algebraic structure is $U(G_q)/C^n(q)$. This is quasi-Hopf algebra and the so called Drinfeld associator characterizes the quasi-associativity.

4.3 Tensor categories

Hopf algebras and related structures do not seem to be quite enough in order to formulate elegantly the construction of S-matrix in TGD framework. A more general structure known as a braided tensor category with left duality and twist operation making the category to a ribbon category is needed. The algebra product μ and co-product Δ must be generalized so that they appear as morphisms $\mu_{A\otimes B\to C}$ and $\Delta_{A\to B\otimes C}$: this gives hopes of describing 3-vertices algebraically. It is not clear whether one can assume single underlying algebra so that objects would correspond to different representations of this algebra or whether one allow even non-isomorphic algebras.

In the tensor category the tensor products of objects and corresponding morphisms belong to the category. In a braided category the objects $U \otimes V$ and $V \otimes U$ are related by a braiding morphism. The notion of braided tensor category appears naturally in topological and conformal quantum field theories and seems to be an appropriate tool also in TGD context. The basic category theoretical notions are discussed in [18] and I have already earlier considered category theory as a possible tool in the construction of quantum TGD and TGD inspired theory of consciousness [E7].

In braided tensor categories one introduces the braiding morphism $c_{V,W} : V\otimes W \rightarrow W\otimes V$, which is closely related to R-matrix. In categories allowing duality arrows with both directions are allowed ad diagrams analogous to pair creation from vacuum are possible. In ribbon categories one introduces also the twist operation θ_V as a morphism of object and the Θ_W satisfies the axiom: $\theta_{V\otimes W} = (\theta_V \otimes \theta_W)c_{W,V}c_{V,W}$. One can also introduce morphisms with arbitrary number of incoming lines and outgoing lines and visualize them as boxes, coupons. Isotopy principle, originally related to link and knot diagrams provides a powerful tool allowing to interpret the basic axioms of ribbon categories in terms of isotopy invariance of the diagrams and to invent theorems by just isotoping.

4.3.1 Categories, functors, natural transformations

Categories [42, 43, 44, 18] are roughly collections of objects A, B, C... and morphisms $f(A \rightarrow B)$ between objects A and B such that decomposition of two morphisms is always defined. Identity morphisms map objects to objects. Examples of categories are open sets of some topological spaces with continuous maps between them acting as morphisms, linear spaces with linear maps between them acting as morphisms, groups with group homomorphisms taking the role of morphisms. Practically any collection of mathematical structures can be regarded as a category. Morphisms can be very general: for instance, partial ordering $a \leq b$ can define a morphism $f(A \rightarrow B)$.

Functors between categories map objects to objects and morphisms to morphisms so that a product of morphisms is mapped to the product of the images and identity morphism is mapped to identity morphism. Functor $F : \mathcal{C} \rightarrow \mathcal{D}$ commutes also with the maps s and b assigning to a morphism $f : V \rightarrow W$ its source $s(f) = V$ and target $b(f) = W$.

A natural transformation between functors F and G from $\mathcal{C} \rightarrow \mathcal{C}'$ is a family of morphisms $\eta(V) : F(V) \rightarrow G(V)$ in $\mathcal{C}'$ indexed by objects V of $\mathcal{C}$ such that for any morphisms $f : V \rightarrow W$ in $\mathcal{C}$, the square

$$\begin{array}{ccc} F(V) & \xrightarrow{\eta(V)} & G(V) \\ \downarrow F(f) & & \downarrow G(f) \\ F(W) & \xrightarrow{\eta(W)} & G(W) \end{array} \tag{42}$$

commutes.

The functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be equivalence of categories if there exists a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ such and natural isomorphisms

$$\eta : id_{\mathcal{D}} \rightarrow FG \text{ and } \theta : GF \rightarrow id_{\mathcal{C}}FG \ .$$

The notion of adjoint functor is a more general notion than equivalence of categories. In this case η and θ are natural transformations but not necessary natural isomorphisms in such a manner that the composite maps

$$F(V) \stackrel{\eta(F(V))}{\longrightarrow} (FGF)(V) \stackrel{F(\theta(V))}{\longrightarrow} F(V)$$

$$G(W) \stackrel{G(\eta(W))}{\longrightarrow} (GFG)(W) \stackrel{\theta(G(W)))}{\longrightarrow} G(W) \tag{43}$$

are identify morphisms for all objects V in $\mathcal{C}$ and W in $\mathcal{D}$.

The product $C = AB$ for objects of categories is defined by the requirement that there exist projection morphisms π_A and π_B from C to A and B and that for any object D and pair of morphisms $f(D \rightarrow A)$ and $g(D \rightarrow B)$ there exist morphism $h(D \rightarrow C)$ such that one has $f = \pi_A h$ and $g = \pi_B h$. Graphically this corresponds to a square diagram in which pairs A,B and C,D correspond to the pairs formed by opposite vertices of the square and arrows DA and DB correspond to morphisms f and g, arrows CA and CB to the morphisms π_A and π_B and the arrow h to the diagonal DC. Examples of product categories are Cartesian products of topological spaces, linear spaces, differentiable manifolds, groups, etc. The tensor products of linear spaces and algebras provides an especially interesting example of product in the recent case. One can define also more advanced concepts such as limits and inverse limits. Also the notions of sheafs, presheafs, and topos are important.

4.3.2 Tensor categories

Let $\mathcal{C}$ be a category. Tensor product $\otimes$ is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ if

1. there is an object $V \otimes W$ associated with any pair (V, W) of objects of $\mathcal{C}$

2. there is an morphism $f \otimes g$ associated with any pair (f, g) of morphisms of $\mathcal{C}$ such that

   $s(f \otimes g) = s(f) \otimes s(g)$ and $b(f \otimes g) = b(f) \otimes b(g)$,

3. if f' and g' are morphisms such that $s(f') = b(f)$ and $s(g') = b(g)$ then

   $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$,

4. $id_{V \otimes W} = id_{W \otimes V}$.

   Any functor with these properties is called tensor product. The tensor product of vector spaces provides the most familiar example of a tensor product functor.

   In figure 4 the general rules for graphical representations of morphisms are given.

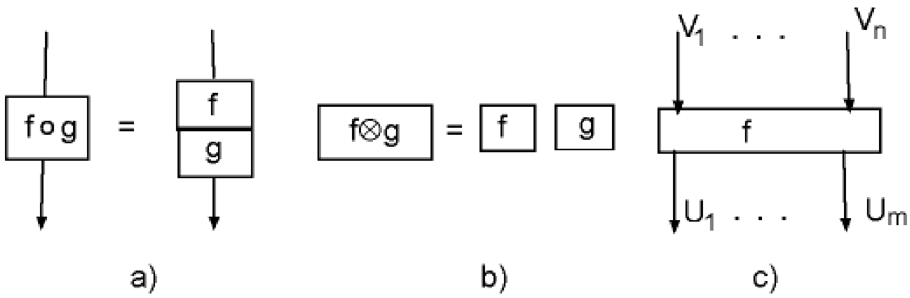


Figure 7: The graphical representation of morphisms. a) $g \circ f$: $V \to W$, b) $f \otimes g$, c) $f : U_1 \otimes ... \otimes U_m \to V_1 \otimes .... \otimes V_n$.

An associativity constraint for the tensor product is a natural isomorphism

$$a : \otimes(\otimes \times id) \to \otimes(id \times \otimes) \ .$$

On basis of general definition of natural isomorphisms (see Eq. 42) one can conclude that for any triple (U, V, W) of objects of $\mathcal{C}$ there exists an isomorphism

$$\begin{array}{ccc} (U \otimes V) \otimes W & \overset{a_{U,V,W}}{\rightarrow} & U \otimes (V \otimes W) \\ \Big\downarrow (f \otimes g) \otimes h & & \Big\downarrow f \otimes (g \otimes h) \\ (U' \otimes V') \otimes W' & \xrightarrow{a_{U',V',W'}} & U' \otimes (V' \otimes W') \end{array} \qquad (44)$$

Associativity constraints satisfies Pentagon Axiom [18] if the following diagrams commutes.

$$
\begin{array}{ccc}
U \otimes (V \otimes W) \otimes X & \xleftarrow[a_{U,V,W} \otimes id_X]{} & ((U \otimes V) \otimes W)) \otimes X \\
 & & \downarrow a_{U \otimes V,W,X} \\
\downarrow a_{U,V \otimes W,X} & & (U \otimes V) \otimes (W \otimes X) \\
 & & \downarrow a_{U,V,W \otimes X} \\
U \otimes ((V \otimes W) \otimes X) & \xrightarrow{id_U \otimes a_{V,W,X}} & U \otimes (V \otimes (W \otimes X))
\end{array}
\tag{45}
$$

Pentagon axiom has been already mentioned while discussing the definition of quasi-Hopf algebras. In figure 4.3.2 are graphical illustrations of associativity morphism $a(U, V, W)$, Triangle Axiom, and Pentagon Axiom are given.

Assume that an object I is fixed in the category. A left unit constraint with respect to I is a natural isomorphism

$$l : \otimes(I \times id) \to id$$

By Eq. 42 this means that for any object V of C there exists an isomorphism

$$l_V : I \otimes V \to V \tag{46}$$

such that

$$
\begin{array}{ccc}
I \otimes V & \xrightarrow{l_V} & V \\
\downarrow id_I \otimes f & & \downarrow f \\
I \otimes V' & \xrightarrow{l_{V'}} & V'
\end{array}
\tag{47}
$$

The right unit constraint $r : \otimes(id \times I) \to id$ can be defined in a completely analogous manner.

Given an associativity constraint a, and left and right unit constraints l, r with respect to an object I, one can say that the Triangle Axiom is satisfies if the triangle

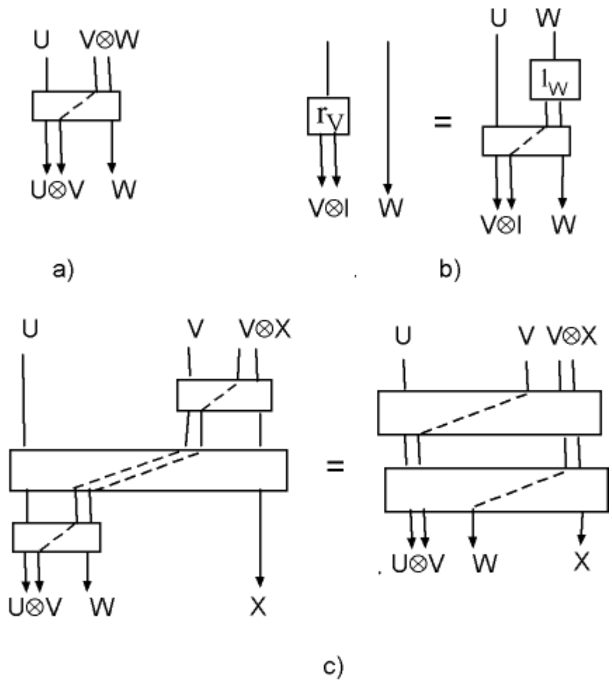


Figure 8: Graphical representations of a) the associativity isomorphism $a_{U,V,W}$, b) Triangle Axiom, c) Pentagon Axiom.

$$\begin{array}{ccc} (V\otimes I)\otimes W & \overset{a_{V,I,W}}{\longrightarrow} & V\otimes(I\otimes W) \\ & {\searrow}^{r_V\otimes id_W} \quad {\swarrow}_{id_W\otimes l_W} & \\ & V\otimes W & \end{array} \tag{48}$$

commutes (see figure 4.3.3).

These ingredients lead allow to define tensor category $(\mathcal{C}, I, a, l, r)$ as a category $\mathcal{C}$ which is equipped with a tensor product $\otimes : \mathcal{C}\times\mathcal{C} \rightarrow \mathcal{C}$ satisfying associativity constraint a, left unit constraint l and right unit constraint r with respect to I, such that Pentagon Axiom and Triangle Axiom are satisfied.

The definition of a tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ involves also additional isomorphisms. $\phi_0 : I \rightarrow F(I)$ satisfies commutative diagrams involving right and left unit constraints l and r. The family of isomorphisms

$$\phi_2(U,V) : F(U) \otimes F(V) \to F(U \otimes V)$$

satisfies a commutative diagram stating that ϕ_2 commutes with associativity constraints. The interested reader can consult [18] for details. One can also define the notions of natural tensor transformation, natural tensor isomorphism, and tensor equivalence between tensor categories by applying the general category theoretical tools.

Keeping track of associativity isomorphisms is obviously a rather heavy burden. Fortunately, it can be shown that one can assign to a tensor category $\mathcal{C}$ a strictly associative (or briefly, strict) tensor category which is tensor equivalent of $\mathcal{C}$.

4.3.3 Braided tensor categories

Braided tensor categories satisfy also commutativity constraint c besides associativity constraint a. Denote by $\tau : \mathcal{C}\times\mathcal{C} \to \mathcal{C}\times\mathcal{C}$ the flip functor defined by $\tau(V,W) = (W,V)$. Commutativity constraint is a natural isomorphism

$$c : \otimes \to \otimes\tau \ .$$

This means that for any pair (V,W) of objects there exists isomorphism

$$c_{V,W} : V \otimes W \to W \otimes V$$

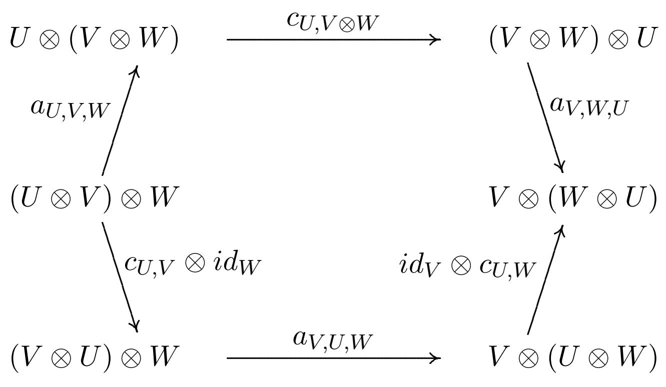
such that the square

$$\begin{array}{ccc} V \otimes W & \xrightarrow{c_{V,W}} & W \otimes V \\ \downarrow f \otimes g & & \downarrow g \otimes f \\ V' \otimes W' & \xrightarrow{c_{V',W'}} & W' \otimes V' \end{array} \qquad (49)$$

commutes.

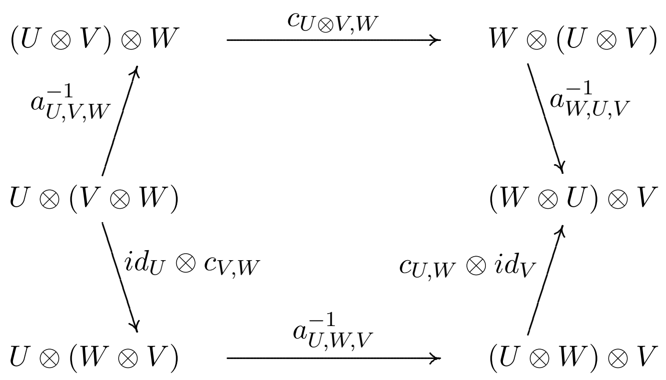
The commutativity constraint satisfies Hexagon Axiom if the two hexagonal diagrams

(H1)



(50)

and (H2)



(51)

commute.

The braiding operation $c_{V,W}$ and the association operation $a(U, V, W)$, and pentagon and hexagon axioms are illustrated in the figure 4.3.3 below.

4.3.4 Duality and tensor categories

The notion of a dual of the finite-dimensional vector space as a space of linear maps from V to field k can lifted to a concept applying for tensor category. A strict (strictly associative) tensor category $(\mathcal{C}, \otimes, I)$ with unit object I is said to possess left duality if for each object V of $\mathcal{C}$ there exists an object $V^\star$ and morphisms

$$b_V : I \to V \otimes V^\star \text{ and } d_V : V^\star \otimes V \to I$$

such that

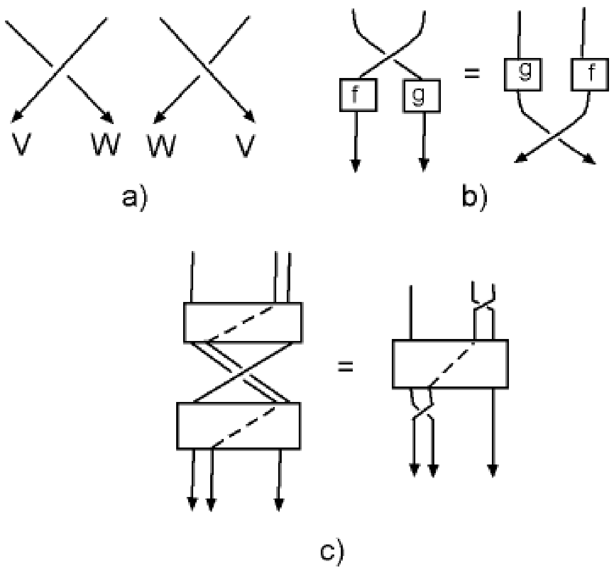


Figure 9: Graphical representations a) of the braiding morphism $c_{V,W}$ and its inverse $c^{-1}_{V,W}$, b) of naturality of $c_{V,W}$, c) of First Hexagon Axiom.

$$(id \otimes d_V)(b_V \otimes id_V) = id_V \ \ and \ \ (d_V \otimes id_{V^\star})(id_{V^\star} \otimes b_V) = id_{V^\star} \ . \tag{52}$$

One can define the transpose of f in terms of b_V and d_V. The idea how this is achieved is obvious from figure 4.3.4.

$$f^\star = (d_V \otimes id_{U^\star})(id_{V^\star} \otimes f \otimes id_{U^\star})(id_{V^\star} \otimes b_U) \ . \tag{53}$$

Also the braiding operation $c_{V^\star,W}$ can be expressed in terms of $c^{-1}_{V,W}$, b_V and d_V by using the isotopy of Fig. 4.3.4:

$$c_{V^\star,W} = (d_V \otimes id_{W\otimes V^\star})(id_{V^\star} \otimes c^{-1}_{V,W} \otimes id_{V^\star})(id_{V^\star\otimes W} \otimes b_V) \ . \tag{54}$$

Drinfeld quantum double can be regarded as a tensor product of Hopf algebra and its dual and in this case one can introduce morphisms ev_H : $H \otimes H^\star \rightarrow k$ defined as $e^i \otimes e_j \rightarrow \delta^i_j$ defining inner product and its inverse $\delta : k \rightarrow H \otimes H$ defined as $1 \rightarrow e^i e_i$, where summation over i is understood.

For categories these morphisms are generalized to morphism d_V from objects V of category to unit object I and b_V from I to object of category. The elements of H and $H^\star$ are described as strands with opposite directions, whereas d_V and b_V correspond to annihilation and creation of strand–anti-strand pair as show in figure 4.3.4.

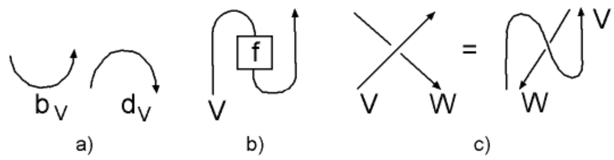


Figure 10: Graphical representations a) of the morphisms b_V and d_V, b) of the transpose $f^\star$, c) of braiding operation $c_{V^\star,W}$ expressed in terms of $c_{V,W}$.

4.3.5 Ribbon categories

According to the definition of [18] ribbon category is a strict braided tensor category $(\mathcal{C},\otimes,I)$ with a left duality with a family of natural morphisms $\theta_V : V \rightarrow V$ indexed by the objects V of $\mathcal{C}$ satisfying the conditions

$$\begin{aligned} \theta_{V\otimes W} &= \theta_V \otimes \theta_W c_{W,V} c_{V,W} \ , \\ \theta_{V^\star} &= (\theta_V)^\star \end{aligned} \tag{55}$$

for all objects V,W of $\mathcal{C}$. The naturality of twist means for for any morphisms $f : V \rightarrow W$ one has $\theta_W f = f\theta_V$. The graphical representation for the axioms and is in Fig. 4.3.5.
The existence of the twist operation provides $\mathcal{C}$ with right duality necessary in order to define trace (see Fig. 4.3.5).

$$\begin{aligned} d'_V &= (id_{V^\star} \otimes \theta_V) c_{V,V^\star} b_V \ , \\ b'_V &= d_V c_{V,V^\star} (\theta_V \otimes id_{V^\star}) \ . \end{aligned} \tag{56}$$

One can define quantum trace for any endomorphisms f of ribbon category:

$$tr_q(f) = d'_V(f \otimes id_{V^\star}) b_V = d_V c_{V,V^\star}(\theta_V f \otimes id_{V^\star}) b_V \ . \tag{57}$$

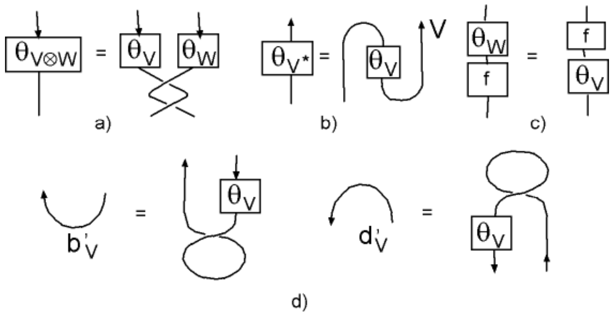


Figure 11: Graphical representations a) of $\theta_{V\otimes W} = \theta_V \otimes \theta_W c_{W,V} c_{V,W}$, b) of $\theta_{V^\star} = (\theta_V)^\star$, c) of $\theta_W f = f\theta_V$, d) of right duality for a ribbon category.

Again the graphical representation is the best manner to understand the definition, see figure 4.3.5. Quantum trace has the basic properties of trace: $tr_q(fg) = tr_q(gf)$, $tr_q(f \otimes g) = tr_q(f)tr_q(g)$, $tr_q(f) = tr_q(f^\star)$. The proof of these properties is easiest using isotropy principle.

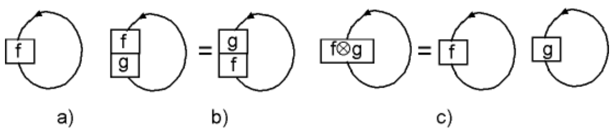


Figure 12: Graphical representations of a) $tr_q(f)$, b) of $tr_q(fg) = tr_q(gf)$, c) of $tr(f \otimes g) = tr(f)tr(g)$.

The quantum dimension of an object V of ribbon category can be defined as the quantum trace for the identity morphism of V: $dim_q(V) = tr_q(id_V) = d'_V b_V$. Quantum dimension is represented as a vacuum bubble. Quantum dimension satisfies the conditions $dim_q(V \otimes W) = dim_q(V)dim_q(W)$ and $dim_q(V) = dim_q(V^\star)$.

A more general definition of ribbon category inspired by the considerations of [19] is obtained by allowing the generalization of morphisms μ and Δ so that they become morphisms $\mu_{A\otimes B\rightarrow C}$ and $\Delta_{C\rightarrow A\otimes B}$ of ribbon category. Graphically the general morphism with arbitrary number of incoming outgoing strands can be represented as a box or "coupon". An important special case of ribbon categories consists of modules over braided Hopf algebras allowing ribbon algebra structure.

5 Axiomatic approach to S-matrix based on the notion of quantum category

This section can be regarded as an attempt of a physicists with some good intuitions and intentions but rather poor algebraic skills to formulate basic axioms about S-matrix in terms of what might be called quantum category. The basic result is an interpretation for the equivalence of loop diagrams with tree diagrams as a consequence of basic algebra and co-algebra axioms generalized to the level of tensor category. The notion of quantum category emerges naturally as a generalization of ribbon category, when algebra product and co-algebra product are interpreted as morphisms between different objects of the ribbon category.

The general picture suggest that the operations Δ and μ generalized to algebra homomorphisms $A \rightarrow B \otimes C$ and $A \otimes B \rightarrow C$ in a tensor category whose objects are either representations of an algebra or even algebras might provide an appropriate mathematical tool for saying something interesting about S-matrix in TGD Universe. These algebras need not necessarily be bi-algebras. In the following it is demonstrated that the equivalence of loop diagrams to tree diagrams follows from suitably generalized bi-algebra axioms. Also the interpretation of various morphisms involved with Hopf algebra structure is discussed.

5.1 Δ and μ and the axioms eliminating loops

The first task is to find a physical interpretation for the basic algebraic operations and how the basic algebra axioms might allow to eliminate loops. The physical interpretation of morphisms Δ and μ as algebra or category morphisms has been already discussed. As already found, the condition that Δ (μ) acts as an algebra (co-algebra) morphism leads to a condition stating that a box graph for 2-particle scattering is equivalent with tree graph. It is interesting to identify the corresponding conditions in the case of self energy loops and vertex corrections.

The condition

$$\mu_{B\otimes C\rightarrow A} \circ \Delta_{A\rightarrow B\otimes C} = K \times id_A \ , \tag{58}$$

where K is a numerical factor, is a natural additional condition stating that a line with a self energy loop is equivalent with a line without the loop. The condition is illustrate in figure 5.1. For the co-commutative tensor algebra $T(V)$ of vector space with $\Delta(x) = 1 \otimes x + x \otimes 1$ one would have $K = 2$ for the generators of $T(V)$. For a product of n generators one has $K = 2^n$.

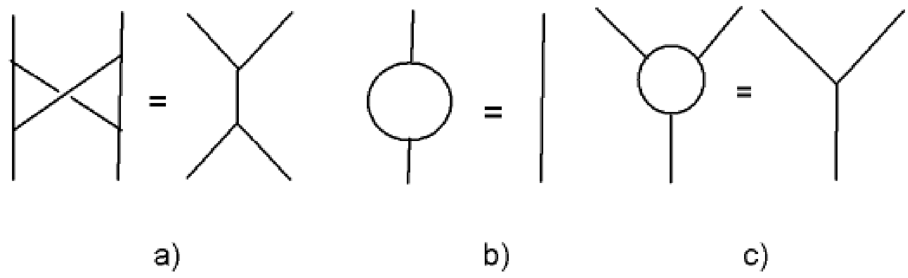


Figure 13: Graphical representations for the conditions a) $(id \otimes \mu \otimes id)(\Delta \otimes \Delta) = \Delta \circ \mu$, b) $\mu_{B\otimes C \to A} \circ \Delta_{A \to B\otimes C} = K \times id_A$, and c) $(\mu \otimes id) \circ (\Delta \otimes id) \circ \Delta = K \times \Delta$.

The condition $\Delta_{A \to B\otimes C} \circ \mu_{B\otimes C \to A} = K \times id_A$ cannot hold true since multiplication is not an irreversible process. If this were the case one could reduce tree diagrams to collections of free propagator lines.

In quantum field theories also vertex corrections are a source of divergences. The requirement that the graph representing a vertex correction is equivalent with a simple tree graph representing a decay gives an additional algebraic condition. For bi-algebras the condition would read

$$(\mu \otimes id) \circ (\Delta \otimes id) \circ \Delta = K\Delta \ , \qquad (59)$$

where K is a simple multiplicative factor. In fact, for the co-commutative tensor algebra $T(V)$ of vector space the left hand side would be $3 \times \Delta(x)$ giving $K = 3$ for generators $T(V)$. The condition is illustrated in figure 5.1.

Using the standard formulas of appendix for quantum groups one finds that in the case of $U_q(sl(2))$ the condition $\mu \circ \Delta(X) = K_X X$, K_X constant, is not true in general. Rather, one has $\mu \circ \Delta(X) = XK_X(q^{H/2} + q^{-H/2}, q^{1/2}, q^{-1/2})$. The action on the vacuum state is however proportional to that of X, being given by $K_X(2,1,1)X$. The function K_X for a given X can be deduced from $\mu \circ \Delta(X_{\pm}) = q^{H/2}X_{\pm} + X_{\pm}q^{-H/2} = X_{\pm}(q^{\pm 1/2} + q^{H/2} + q^{-H/2})$. The eigen states of Cartan algebra generators are expected to be eigen states of $\mu \circ \Delta$ also in the case of a general quantum group. $\mu \circ \Delta$ is analogous to a single particle operator like kinetic energy and its action on multi-particle state is a sum over all tensor factors with $\mu \circ \Delta$ applied to each of them. For eigen states of $\mu \circ \Delta$ the projective equivalence of loop diagrams with tree diagrams would make sense.

Since self energy loops, vertex corrections, and box diagrams represent the basic divergences of renormalizable quantum field theories, these axioms raise the hope that the basic infinities of quantum field theories could be eliminated by the basic axioms for the morphisms of quantum category.

There are also morphisms related to the topology changes in which the 3-surface remains connected. For instance, processes in which the number of boundary components can change could be of special relevance if the family replication phenomenon reduces to the boundary topology. Also 3-topology can change. The experience with topological quantum field theories [34], stimulates the hope that the braid group representations of the topological invariants of 3-topology might be of help in the construction of S-matrix.

The equivalence of loop diagrams with tree diagrams must have algebraic formulation using the language of standard quantum field theory. In the third section it was indeed found that thanks to the presence of the emission of vacuons, the equivalence of loop diagrams with tree diagrams corresponds to the vanishing of loop corrections in the standard quantum field theory framework. Furthermore, the non-cocommutative Hopf algebra of Feynman diagrams discussed in [45] becomes co-commutative when the loop corrections vanish so that TGD program indeed has an elegant algebraic formulation also in the standard framework.

5.2 The physical interpretation of non-trivial braiding and quasi-associativity

The exchange of the tensor factors by braiding could also correspond to a physically non-trivial but unitary operation as it indeed does in anyon physics [46, 47]. What would differentiate between elementary particles and anyons would be the non-triviality of the super-canonical and Super Kac-Moody conformal central extensions which have the same origin (addition of a multiplication by a multiple of the Hamiltonian of a canonical transformation to the action of isometry generator). The proposed interpretation of braiding acting in the complex plane in which the conformal weights of the elements of the super-canonical algebra represent punctures justifies the non-triviality. Hexagon Axioms would state that two generalized Feynman diagrams involving exchanges, dissociations and re-associations are equivalent.

An interesting question is whether the association $(A,B) \to (A \otimes B)$ could be interpreted as a formation of bound state entanglement between A and B. A possible space-time correlate for association is topological condensation of A and B to the same space-time sheet. Association would be trivial if all particles are at same space-time sheet X^4 but non-trivial if some subset of particles condense at an intermediate space-time sheet Y^4 condensing in turn at X^4.

Be as it may, association isomorphisms $a_{A,B,C}$ would state that the state space obtained by binding A with bound bound states $(B \otimes C)$ is unitarily related with the state space obtained by binding $(A \otimes B)$ bound states with C. With this interpretation Pentagon axiom would state that two generalized Feynman diagrams depicted in figure 4.3.2 leading from initial to final to final state by dissociation and reassociation are equivalent.

5.3 Generalizing the notion of bi-algebra structures at the level of configuration space

Configuration space of 3-surfaces decomposes into sectors corresponding to different 3-topologies. Also other signatures might be involved and I have proposed that the sectors are characterized by the collection of p-adic primes labelling space-time sheets of the 3-surface and that a given space-time surface could be characterized by an infinite prime or integer. The general problem is to continue various geometric structures from a given sector A of configuration space to other sector B.

An especially interesting special case corresponds to a continuation from 1-particle sector to two-particle sector or vice versa and corresponds to TGD variant of 3-vertex. All these continuations involve the imbedding of a structure associated with the sector A to a structure associated with sector B. For the continuation from 1-particle sector to 2-particle sector the map is analogous to co-algebra homomorphism Δ. For the reverse continuation it is analogous to the algebra product μ. Now however one does not have maps $\Delta : A \to A \otimes A$ and $\mu : A \otimes A \to A$ but $\Delta : A \to B \otimes C$ and $\mu : B \otimes C \to A$ unless the algebras are isomorphic. $\mu \circ \Delta = id$ should hold true as an additional condition but $\Delta \circ \mu = id$ cannot hold true since product maps many pairs to the same element.

5.3.1 Continuation of the configuration space spinor structure

The basic example of a structure to be continued is configuration space spinor structure. Configuration space spinor fields in different sectors should be related to each other. The isometry generators and gamma matrices of configuration space span a super-canonical algebra. The continuation requires that the super algebra basis of different sectors are related. Also vacua must be related. Isometry generators correspond to bosonic generators of the super-canonical algebra. There is also a natural extension of the super-canonical algebra defined by the Poisson structure of the configuration space.

This view suggests that in the first approximation one could see the construction of S-matrix as following process.

1. Incoming/outgoing states correspond to positive/negative energy states localized to the sectors of configuration space with fixed 3-topologies.

2. In order to construct an S-matrix matrix element between two states localized in sectors A and B, one must continue the state localized in A to B or vice versa and calculate overlap. The continuation involves a sequence of morphisms mapping various structures between sectors. In particular, topological transformations describing particle decay and fusion are possible so that the analogs of product μ and co-product Δ are involved. The construction of three-manifold topological invariants [34] in topological quantum field theories provides concrete ideas about how to proceed.

3. The S-matrix element describing a particular transition can be expressed as any path leading from the sector A to B or vice versa. There is a huge symmetry very much analogous to the independence of the final result of the analytic continuation on the path chosen since generalized Feynman graphs allow all moves changing intermediate topologies so that initial and final 3-topologies are same. Generalized conformal invariance probably also poses restrictions on possible paths of continuation. In the path integral approach one would have simply sum over all these equivalent paths and thus encounter the fundamental difficulties related to the infinite-dimensional integration.

4. Quantum classical correspondence suggests that the continuation operation has a space-time correlate. That is, the absolute minimum of Kähler action going through the initial and final 3-sheets defines a sequence of transitions changing the topology of 3-sheet. The localization to a particular sector of course selects particular absolute minimum. There are two possible interpretations. Either the continuation from A is not possible to all possible sectors but only to those

with 3-topologies appearing in X^4, or the absolute minimum represents some kind of minimal continuation involving minimum amount of calculational labor.

5. Quantum classical correspondence and the possibility to represent the rows of S-matrix as zero energy quantum states suggests that the paths for continuation can be also represented at the space-time level, perhaps in terms of braided join along boundaries bonds connecting two light like 3-surfaces representing the initial and final states of particle reaction. Since light like 3-surfaces are metrically two-dimensional and allow conformal invariance, this suggests a connection with braid diagrams in the sense that it should be possible to regard the paths connecting sectors of configuration space consisting of unions of disjoint 3-surfaces (corresponding interacting 4-surfaces are connected) as generalized braids for which also decay and fusion for the strands of braid are possible. Quantum algebra structure and effective metric 2-dimensionality of the light like 3-surfaces suggests different braidings for join along boundaries bonds connecting boundaries of 3-surfaces define non-equivalent 3-surfaces.

5.3.2 Co-multiplication and second quantized induced spinor fields

At the microscopic level the construction of S-matrix reduces to understanding what happens for the classical spinor fields in a vertex, which corresponds to an incoming 3-surface A decaying to two outgoing 3-surfaces B and C. At the classical level incoming spinor field A develops into a spinor fields B and C expressible as linear combinations of appropriate spinor basis. At quantum level one must understand how the Fock space defined by the incoming spinor fields of A is mapped to the tensor product of Fock spaces of B and C. The idea about the possible importance of co-algebras came with the realization that this mapping is obviously is very much like a co-product. Co-algebras and bi-algebras possessing both algebra and co-algebra structure indeed suggest a general approach giving hopes of understanding how Feynman diagrammatics generalizes to TGD framework.

The first guess is that fermionic oscillator operators are mapped by the imbedding Δ to a superposition of operators $a^{\dagger}_{Bn} \otimes Id_C$ and $Id_B \otimes a^{\dagger}_{Cn}$ with obvious formulas for Hermitian conjugates. Δ induces the mapping of higher Fock states and the construction of S-matrix should reduce to the construction of this map.

Δ is analogous to the definition for co-product operation although there is also an obvious difference due to the fact that Δ imbeds algebra A to $B \otimes C$ rather than to $A \otimes A$. Only in the case that the algebras are isomorphic, the situation reduces to that for Hopf algebras. Category theoretical approach however allows to consider a more general situation in which Δ is a morphism in the category of Fock algebras associated with 3-surfaces.

Δ preserves fermion number and should respect Fock algebra structure, in particular commute with the anti-commutation relations of fermionic oscillator operators. The basis of fermionic oscillator operators would naturally correspond to fermionic super-canonical generators in turn defining configuration space gamma matrices.

Since any leg can be regarded as incoming leg, strong consistency conditions result on the coefficients in the expression

$$\Delta(a^{\dagger}_{An}) \quad = \quad C(A,B)_n{}^m a^{\dagger}_{Bm} \otimes Id_C + C(A,C)_n{}^m Id_B \otimes a^{\dagger}_{Cm} \qquad (60)$$

by forming the cyclic permutations in A, B, C. This option corresponds to the co-commutative situation and quantum group structure. If identity matrices are replaced with something more general, co-product becomes non-cocommutative.

5.4 Ribbon category as a fundamental structure?

There exists a generalization of the braided tensor category inspired by the axiomatic approach to topological quantum field theories which seems to almost catch the proposed mathematical requirements. This category is also called ribbon category [33] but in more general sense than it is defined in [18].

One adds to the tangle diagrams (braid diagrams with both directions of strands and possibility of strand–anti-strand annihilation) also "coupons", which are boxes representing morphisms with arbitrary numbers of incoming and outgoing strands. As a special case 3-particle vertices are obtained. The strands correspond to representations of a fixed Hopf algebra H.

In the recent case it would seem safest to postulate that strands correspond to algebras, which can be different because of the potential dependence of the details of Fock algebra on 3-topology and other properties of 3-surface. For instance, configuration space metric defined by anti-commutators of the gamma matrices is degenerate for vacuum extremals so that the infinite Clifford algebra is definitely "smaller" than for surfaces with $D \geq 3$-dimensional CP_2 projection.

One might feel that the full ribbon algebra is an un-necessary luxury since only 3-particle vertices are needed since higher vertices describing decays of 3-surfaces can be decomposed to 3-vertices in the generic case. On the other hand, many-sheeted space-time and p-adic fractality suggest that coupons with arbitrary number of incoming and outgoing strands are needed in order to obtain the p-adic hierarchy of length scale dependent theories.

The situation would be the same as in the effective quantum field theories involving arbitrarily high vertices and would require what might be called universal algebra allowing n-ary multiplications and co-multiplications rather than only binary ones. Also strands within strands hierarchy is strongly suggestive and would require a fractal generalization of the ribbon algebra. Note that associativity and commutativity conditions for morphisms which more than three incoming and outgoing lines would force to generalize the notion of R-matrix and would bring in conditions stating that more complex loop diagrams are equivalent with tree diagrams.

5.5 Minimal models and TGD

Quaternion conformal invariance with non-vanishing c and k for anyons is highly attractive option and minimal super-conformal field theories attractive candidate since they describe critical systems and TGD Universe is indeed a quantum critical system.

5.5.1 Rational conformal field theories and TGD

The highest weight representations of Virasoro algebra are known as Verma modules containing besides the ground state with conformal weight Δ the states generated by Virasoro generators L_n, $n \geq 0$. For some values of Δ Verma module contains states with conformal weight $\Delta + l$ annihilated by Virasoro generators L_n, $n \geq 1$. In this case the number of primary fields is reduced since Virasoro algebra acts as a gauge algebra. The conformal weights Δ of the Verma modules allowing null states are given by the Kac formula

$$\begin{aligned}
\Delta_{mm'} &= \Delta_0 + \frac{1}{4}(\alpha_+ m + \alpha_- m')^2 \ , \quad m, m' \in \{1, 2...\} \ , && (61)\\
\Delta_0 &= \frac{1}{24}(c-1) \ , \\
\alpha_{\pm} &= \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \ . && (62)
\end{aligned}$$

The descendants $\prod_{n\geq 1} L_n^{k_n}|\Delta\rangle$ annihilated by L_n, $n>0$, have conformal weights at level $l=\sum_n nk_n = mm'$.

In the general case the operator products of primary fields satisfying these conditions form an algebra spanned by infinitely many primary fields. The situation changes if the central charge c satisfies the condition

$$c = 1-\frac{6(p'-p)^2}{pp'} \ , \tag{63}$$

where p and p' are mutually prime positive integers satisfying $p<p'$. In this case the Kac weights are rational

$$\Delta_{m,m'} = \frac{(mp'-m'p)^2-(p'-p)^2)}{4pp'} \ , \ 0<m<p \ , \ 0<m'<p' \ . \tag{64}$$

Obviously, the number of primary fields is finite. This option does not seem to be realistic in TGD framework were super-conformal invariance is realized.

For $N=1$ super-conformal invariance the unitary representations have central extension and conformal weights given by

$$\begin{aligned} c &= \frac{3}{2}(1-\frac{8}{m(m+2)}) \ , \\ \Delta_{p,q}(NS) &= \frac{\left[(m+2)p)-mq)\right]^2-4}{8m(m+2)} \ , \ 0\leq p\leq m \ , \ 1\leq q\leq m+2 \ . \end{aligned} \tag{65}$$

For Ramond representations the conformal weights are

$$\Delta_{p,q}(R) = \Delta(NS)+\frac{1}{16} \ . \tag{66}$$

The states with vanishing conformal weights correspond to light elementary particles and the states with $p=q$ have vanishing conformal weight in NS sector. Also this option is non-realistic since in TGD framework super-generators carry fermion number so that G cannot be a Hermitian operator.

$N=2$ super-conformal algebra is the most interesting one from TGD point of view since it involves also a bosonic $U(1)$ charge identifiable as fermion number and $G^{\pm}(z)$ indeed carry $U(1)$ charge[1]. Hence one has $N = 2$ super-conformal algebra is generated by the energy momentum tensor $T(z)$, $U(1)$ current $J(z)$, and super generators $G^{\pm}(z)$. $U(1)$ current would correspond to fermion number and super generators would involve contraction of covariantly constant neutrino spinor with second quantized induced spinor field. The further facts that $N = 2$ algebra is associated naturally with Kähler geometry, that the partition functions associated with $N = 2$ super-conformal representations are modular invariant, and that $N = 2$ algebra defines so called chiral ring defining a topological quantum field theory [19], lend further support for the belief that $N = 2$ super-conformal algebra acts in super-canonical degrees of freedom.

The values of c and conformal weights for $N = 2$ super-conformal field theories are given by

$$\begin{aligned} c &= \frac{3k}{k+2} \ , \\ \Delta_{l,m}(NS) &= \frac{l(l+2)-m^2}{4(k+2)} \ , \ \ l = 0,1,...,k \ , \\ q_m &= \frac{m}{k+2} \ , \ \ m = -l,-l+2,....,l-2,l \ . \end{aligned} \tag{67}$$

q_m is the fractional value of the $U(1)$ charge, which would now correspond to a fractional fermion number. For $k = 1$ one would have $q = 0, 1/3, -1/3$, which brings in mind anyons. $\Delta_{l=0,m=0} = 0$ state would correspond to a massless state with a vanishing fermion number. Note that $SU(2)_k$ Wess-Zumino model has the same value of c but different conformal weights. More information about conformal algebras can be found from the appendix of [19].

For Ramond representation $L_0 - c/24$ or equivalently G_0 must annihilate the massless states. This occurs for $\Delta = c/24$ giving the condition $k = 2\left[l(l+2) - m^2\right]$ (note that k must be even and that $(k,l,m) = (4,1,1)$ is the simplest non-trivial solution to the condition). Note the appearance of a fractional vacuum fermion number $q_{vac} = \pm c/12 = \pm k/4(k+2)$. I have proposed that NS and Ramond algebras could combine to a larger algebra containing also lepto-quark type generators.

Quaternion conformal invariance [B4] encourages to consider the possibility of super-symmetrizing also spin and electro-weak spin of fermions. In

[1] I realized that TGD super-conformal algebra corresponds to $N = 2$ algebra while writing this and proposed it earlier as a generalization of super-conformal algebra!

this case the conformal algebra would extend to a direct sum of Ramond and NS $N = 8$ algebras associated with quarks and leptons. This algebra in turn extends to a larger algebra if lepto-quark generators acting as half odd-integer Virasoro generators are allowed. The algebra would contain spin and electro-weak spin as fermionic indices. Poincare and color Kac-Moody generators would act as symplectically extended isometry generators on configuration space Hamiltonians expressible in terms of Hamiltonians of $X^3_l \times CP_2$. Electro-weak and color Kac-Moody currents have conformal weight $h = 1$ whereas T and G have conformal weights $h = 2$ and $h = 3/2$.

The experience with $N = 4$ super-conformal invariance suggests that the extended algebra requires the inclusion of also second quantized induced spinor fields with $h = 1/2$ and their super-partners with $h = 0$ and realized as fermion-antifermion bilinears. Since G and Ψ are labelled by 2×4 spinor indices, super-partners would correspond to $2\times(3+1) = 8$ massless electro-weak gauge boson states with polarization included. Their inclusion would make the theory highly predictive since induced spinor and electro-weak fields are the fundamental fields in TGD.

In TGD framework both quark and lepton numbers correspond to NS and Ramond type representations, which in conformal field theories can be assigned to the topologies of complex plane and cylinder. This would suggest that a given three-surface allows either NS or Ramond representation and is either leptonic or quark like but one must be very cautious with this kind of conclusion. Interestingly, NS and Ramond type representations allow a symmetry acting as a spectral flow in the indices of the generators and transforming NS and Ramond type representations continuously to each other [19]. The flow acts as

$$\begin{aligned} L_n &\rightarrow L_n + \alpha J_n + \frac{c}{6}\alpha^2\delta_{n,0} \\ J_n &\rightarrow J_n + \frac{c}{3}\alpha\delta_{n,0} \ , \\ G^{\pm}_n &\rightarrow G^{\pm}_{n\pm\alpha} \ . \end{aligned} \tag{68}$$

The choice $\alpha = \pm 1/2$ transforms NS representation to Ramond representation. The idea that leptons could be transformed to quarks in a continuous manner does not sound attractive in TGD framework. Note that the action of Super Kac-Moody Virasoro algebra in the space of super-canonical conformal weights can be interpreted as a spectral flow.

5.5.2 Co-product for Super Kac-Moody and Super Virasoro algebras

By the previous considerations the quantized conformal weights z_1, z_2, z_3 of super-canonical generators defining punctures of 2-surface should correspond to line punctures of 3-surface. One cannot avoid the thought that these line punctures should meet at single point so that three-vertex would have also quantum field theoretical interpretation.

Each point z_k corresponds to its own Virasoro algebra $V_k = \{L_n^{z_k)}\}$ and Kac-Moody algebra $J_k = \{J_n^{z_k)}\}$ defined by Laurent series of $T(z)$ and $J(z)$ at z_k. Also super-generators are involved. To minimize notational labor denote by $X_n^{z_k)}$, $k = 1, 2, 3$ the generators in question.

The co-algebra product for Super-Virasoro and Super-Kac-Moody involves in the case of fusion $A_1 \otimes A_2 \rightarrow A_3$ a co-algebra product assigning to the generators $X_n^{z_3)}$ direct sum of generators of $X_k^{z_1)}$ and $X_l^{z_2)}$. The most straightforward approach is to express the generators $X_n^{z_3)}$ in terms of generators $X_k^{z_1)}$ and $X_l^{z_2)}$. This is achieved by using the expressions for generators as residy integrals of energy momentum tensor and Kac Moody currents. For Virasoro generators this is carried out explicitly in [19]. The resulting co-product conserves the value of central extension whereas for the naive co-product this would not be the case. Obviously, the geometric co-product does not conserve conformal weight.

6 Is renormalization invariance a gauge symmetry or a symmetry at fixed point?

The notion of renormalization group invariance has a slightly different content in quantum field theories and in TGD framework. This allows to understand what goes wrong with ordinary Feynman diagrammatics and why the renormalization procedure combined with experimental input still works.

6.1 How renormalization group invariance and p-adic topology might relate?

Renormalization group invariance in the standard sense would suggest that propagators, vertices, and in fact all generalized Feynman diagrams D, have a vanishing logarithmic derivative with respect to the parameter λ:

$$\lambda\frac{d}{d\lambda}D \;=\; 0 \; . \qquad (69)$$

All parameters are functions of λ and the interpretation would be as a gauge symmetry. The equation holds true for on mass shell external momenta. In TGD context this equation is not expected to hold true generally and only determines the values of λ, which correspond to the fixed points of renormalization group evolution at which effective gauge symmetry property holds true. The critical of λ should correspond to p-adic length scales. This interpretation allows to assign different values of λ to different particles.

p-Adicization suggests another manner to see the situation. The requirement that $X \equiv \lambda^2$ is well defined both in the real and p-adic sense implies that λ^2 is a rational number. This means a discretization. p-Adicization poses a cutoff so that X must be integer. The prediction that these values correspond to stationary points of $\lambda d/d\lambda$ in the real context is a strong constraint on the algebraic structure. It is also expected to lead to the quantization of masses and various dimensionless coupling constants. Formally D as a function of a discrete variable $X = N$ is pseudo constant in the sense that the derivative vanishes everywhere unlike in the real context and quantum criticality holds true in a stronger sense.

The primes p dividing $X = N$ are in a preferred position since the p-adic norm of X is in this case smaller than one. This is in fact necessary for the series $\sum_{n>1} D^{-2n}$, $D = (p^2 - m^2)/\lambda^2$ appearing in the expression for the propagator to converge p-adically when $p^2 - m^2$ has p-adic norm equal to unity. Thus p-adic topologies R_p for which p divides X are favored. Unless this is the case $p^2 - m^2$ must have p-adic norm smaller than one and the particle would be non-relativistic in the p-adic sense.

6.2 How generalized Feynman diagrams relate to tangles with chords?

The work of Vassilev, Kontsevich and other mathematicians related to the universal knot invariant of finite type [18] generalizes the notion of knot invariant so that it makes sense also for singular knots having double points as self-intersections. The invariant of the singular knot is obtained by replacing singular points with differences of two possible crossings resulting in de-singularization so that ordinary knots result.

1. *Tangles with chords as a tool to produce universal knot invariant*

The construction of knot invariants for singular knots involves chord diagrams as an auxiliary tool. Chord diagrams are circles containing m pairs of points connected by chords. The immersions of circles defining chord diagrams to R^3 define singular knots resulting when the chords are contracted effectively to points by mapping their end points to a single point of R^3. Vacuum Feynman diagrams can be seen as the physical analogs of the chord diagrams associated with circle.

Chords can be attached also to a tangles, whose endpoints defining their boundaries are fixed and instead of circle the strands of the tangle contain the 3-vertices. The physical analogy of chords would be as particle exchanges allowing only 3-vertices at strands of the tangle but not elsewhere. Tensor product and concatenation of the tangles are well defined operations also in the presence of chords. Chord diagrams associated with tangles allow Hopf algebra structure graded by the number c of chords (c relates to the number L of loops by $c = L - 1$ for chord diagrams associated with circles). The tangles A and B such that the source of A is target of B can be fused to tangle $A \circ B$ and this induces a product $\mu : A \otimes B \rightarrow A \circ B$ of the chord diagrams. The co-commutative co-product $\Delta(D)$ of chorded tangle D is obtained by forming the sum of the tensor products $D' \otimes D''$ of all sub-diagrams $D' \subset D$ and their complements D''.

2. How tangles with chords relate to generalized Feynman diagrams?

It is interesting to see how tangles with chords could relate to generalized Feynman diagrams whose symmetries should code the basic axioms of the underlying algebraic structure expressing in turn quantum criticality.

1. The structure of generalized Feynman graphs differs from that of tangles with chords in the sense that also internal 3-vertices are allowed. The imbedding of the tangles to singular braids does not make sense since the mapping of the end points of the chords of a tree diagram to single point would transform tree diagrams to diagrams having no 3-vertices. Neither does the difference $K_+ - K_-$ of knot diagrams have any obvious interpretation in terms of Feynman diagrams. This does not however exclude the possibility of mapping of the diagrams to*non-singular* braids in R^3.

2. The expression for quantum criticality as a symmetry is the free motion of the end points of the chords along the lines of the tangle. The end points of the chord can be made to co-incide in the final situation if the tangle contains a path connecting the points. In this case a self energy loop with coinciding end points results. It is possible to continue the

motion of the second end of the loop so that a 3-vertex with a line containing at its end a bubble necessarily representing vacuum state results, and must correspond to emission of vacuon and can thus be eliminated. Also higher vertices can be decomposed to 3-vertices using the same trick. When the end points of the chord cannot be connected by a path contained by the tangle, chord cannot be eliminated. All loops can be eliminated in this manner so that the outcome is a tree diagram.

3. If knotting would matter, one would obtain an infinite number of non-equivalent diagrams since both internal and external lines could be knotted. It seems that only braiding of the external lines can be allowed. Note that braiding is basically linking and thus represents a "many-particle phenomenon" whereas knotting is "single particle phenomenon" (sub-manifolds X and Y with dimensions $n > 0$ and $D-n-1$ can become linked whereas knotting occurs for sub-manifolds of dimension $D-2$).

4. The possibility to move freely the end points of the chords along lines means that the moves are more general than those induced by orientation preserving maps acting on the lines preserving the ordering of the points associated with the lines so that it is not possible to get through vertices. For instance, a non-planar diagram describing a vertex correction can be transformed to a corresponding planar diagram. This might relate to the fact that p-adic numbers are not well-ordered and that the diagrams should make sense also in the p-adic context.

6.3 Do standard Feynman diagrammatics and TGD inspired diagrammatics express the same symmetry?

The computational formalism of quantum field theories has enjoyed an enormous success, and this forces to ask whether the Feynman rules of renormalized quantum field theory provide a very cumbersome manner to express essentially the same great idea than TGD view about Feynman diagrams tries to do. If so, the equivalence of loop diagrams with tree diagrams must have an algebraic formulation using the language of the standard quantum field theory. In will be indeed found that, thanks to the presence of the emission of vacuons, the equivalence of loop diagrams with tree diagrams corresponds to the vanishing of loop corrections in the standard quantum field theory framework (vacuum lines have vacuum extremals as classical space-time correlates). Thus there is a transition from TGD based notion of Feynman diagram to that of ordinary Feynman diagram. One might say that generalized diagrams relate to Feynman diagrams like an integral to the integrand which is constant.

The following considerations support the view that there is a delicate difference between the philosophies behind the two approaches. In the standard quantum field theory approach scalings act as gauge symmetries whereas in TGD they are analogous to isometries at a fixed point by quantum criticality. p-Adic length scales define the preferred length scales associated with the fixed points. An effective reduction to gauge symmetry however occurs at the fixed points.

1. The Hopf and Lie algebras associated with Feynman diagrams

The work of Connes and Kreimer [45] provides support for the vision declared above. Connes and Kreimer Connes and Kreiman accept quantum field theory as such and generalize the Hopf algebra and Lie algebra structures associated with diagrams defined by tangles with chords so that they apply to Feynman diagrams. The enveloping algebra of Lie-algebra of Feynman diagrams can be identified as the dual of the corresponding Hopf algebra and assigns numerical expressions to Feynman graphs. The Lie algebra exponentiates to an infinite-dimensional group G. TGD approach can be seen as a diametrical opposite of this approach: the idea is to replace Feynman diagrammatics with chorded tangle diagrammatics with quantum criticality as the fundamental symmetry allowing to eliminate loops.

The outcome is a systematic and very elegant representation of the dimensional regularization procedure. The complex sphere S^2 defined by a complexified space-time dimension D emerges naturally since the dimensional regularization involves a small complex curve $D = 4 + \epsilon(z)$ surrounding the point $D = 4$. The curve divides S^2 to two disjoint parts C_+ and C_-. The dependence of the Feynman graph on complexified dimension defines a map $z \rightarrow \gamma(z)$ from S^2 to the group G. The Birkhoff decomposition $\gamma(z) = \gamma_-^{-1}(z)\gamma_+(z)$ such that $\gamma_\pm(z)$ is holomorphic in $C_\pm$ allows to extract the finite part of the Feynmann diagram as the value of $\gamma_+(z)$ at $z = D = 4$.

Only one-particle irreducible (1PI) Feynman graphs (not decomposing to two disconnected parts when single line is removed) relevant for the effective action are considered. The commutative product for Feynman graphs is simply the union of graphs whereas co-product involves the decomposition of the graph in various manners to a sub-graph D_1 and its complement D_1^c and summing over all possible tensor products $D \otimes D_1^c$. Graph and complement are allowed to meet only through two or three lines, which relates to the assumption that no higher vertices are allowed in the model considered. A somewhat analogous decomposition appears in Bogoliubov's recursion formula and in Zimmermann's general solution to it using the "forest formula" [49] and giving renormalized quantities without any need for regularization. Co-product is not co-commutative.

The Lie-algebra action in the space of 1PI Feynman diagrams has interpretation in terms of insertions and eliminations for Feynman diagrams, the Lie bracket being computable from the insertions of one graph to another one and vice versa. Lie-algebra generators corresponds to linear complex valued functionals $f(X)$ in the space of 1PI Feynman diagrams with obvious product and $\star$ operation $f\star g(X) = \mu\circ(f\otimes g)\Delta(X))$ for the dual algebra is used to define associative product in turn defining the Lie algebra commutator. Elimination means the replacement of a sub-graph with a single vertex graph having the same external lines as the original graph. Insertion is the inverse of this procedure. For instance, a bare vertex can be replaced with a vertex containing some radiative corrections.

2. The Lie-algebra of Feynman diagrams annihilates Green's functions at quantum criticality in TGD Universe

In TGD framework the non-cocommutative Hopf algebra of Feynman diagrams becomes effectively co-commutative at quantum critical point since loop corrections vanish. Also the action of the scaling transformation realized infinitesimally as $\lambda d/d\lambda$ operation leaves the expressions associated with Feynman diagrams invariant only for the *critical* values of λ. In standard renormalization theory the action of the scaling transformation would be a gauge symmetry for *all* values of renormalization parameter λ.

Quite generally, there are two basic manners to realize gauge invariance. One can sum over all gauge equivalent configurations and form an average or one can perform a gauge fixing. The standard renormalization group invariant quantum field theory based on the functional integral and summation over Feynman diagrams could be seen as an attempt to produce renormalization group gauge invariance by simply taking an average over all gauge equivalent Feynman diagrams by using a gauge invariant integration measure. Divergence difficulties could be seen as being due to the fact that this integration measure is un-normalized and gives infinite values so that one must introduce infinite renormalization constants. In TGD based diagrammatics the effective gauge invariance would be realized by picking just single tree diagram from the equivalence class of physically equivalent diagrams, or if standard Feynmann diagrammatics is used, by the vanishing of loop corrections.

6.4 How p-adic coupling constant evolution is implied by the vanishing of loops?

Consider next how the equivalence of loop diagrams with tree diagrams can be consistent with the vanishing of loop corrections and how this equivalence could fix the coupling constant evolution.

1. What is new as compared to quantum field theory is that algebraic approach predicts the possibility of vacuum lines to which one assigns an identity operator and vanishing momentum and super-canonical conformal weight. This means that the amplitude associated with a given loop diagram contains terms associated with simpler diagrams with some internal propagator lines eliminated. The equivalence of loop diagrams with tree diagrams would mean that the conservation of conformal weight in vertices allows only identity operator in the loop propagator lines describing exchanged particles so that the reduction to tree diagrams occurs. The contributions with non-trivial propagators should vanish by the requirement that the allowed conformal weights correspond to the zeros of relevant polyzetas. The reduction of loop diagrams to tree diagrams uniquely fix evolution of the vertices as a function λ and thus of p-adic length scale.

2. Consider first the conditions from the reduction of loops with vacuum exchanges to tree diagrams. In the case of a self energy bubble defined by 3-vertex the reduced part of the diagram would correspond to an insertion of a term $V_{vac}GV_{vac}$, where V_{vac} is the vertex for the emission of the vacuum line and G is the propagator. The reduction to a tree diagram gives the condition

$$V_{vac}GV_{vac} = 1 \ . \tag{70}$$

The reduction of 3-vertex with triangle to tree-vertex gives

$$(V_{vac}G \otimes V_{vac}G)V = V \ . \tag{71}$$

The reduction of the box diagram to tree diagram gives

$$(1 \otimes G \otimes 1)(V_{vac}G \otimes 1)V \otimes (1 \otimes V_{vac}G)V = (1 \otimes G \otimes 1)V \otimes V \ . \tag{72}$$

Note that V is analogous to the co-multiplication Δ and $V_{vac}GV_{vac}$ is analogous to co-multiplication $H \rightarrow H \otimes k$ defined as the inverse of the multiplication of algebra element by the field k with respect to which algebra is a linear space.

3. The vertices depend on the conformal weights of the particles emanating from the vertex (besides momenta and other quantum numbers) so that for three vertex one has $V_3 = V(z_1, z_2, z_3)$. The natural guess is that conformal weights or at least their imaginary parts, which are highly analogous to ordinary momenta, are conserved in the vertices and that the conformal weights $z_1, z_2, ..., z_k$ emanating from the vertex correspond to zeros of $\zeta(z_1, z_2, ...z_k)$. As already found, the conservation of the entire conformal weight allows non-vanishing tree diagrams in the case of three-vertices so that this condition looks sensible.

4. Loop corrections associated with the vertices should vanish. This is achieved if each external 3-vertex associated with the loop gives in the loop integration a factor proportional to $\zeta(z_1, z_2)$, where z_i are the conformal weights associated with the internal lines of the vertex. For n-vertices $\zeta(z_1, z_2, ..., z_{n-1})$ should result in the loop integration.

   A stronger hypothesis is that all loop integrals, say an arbitrary vacuum bubble containing an arbitrary number of propagator lines emanating from the points of a circle and connected to the points inside the circle assigns a factor $\zeta(z_1, z_2, ..z_k)$ with each external k-vertex. Effectively this would mean a separation of variables in the conformal degrees of freedom associated with the external vertices. Since essentially topological degrees of freedom are in question, one might hope that this is possible.

The conditions are very stringent and might allow only three-vertices in accordance with the assumption that the Hopf algebra based category describing vertices relies on binary algebraic operations. The condition would relate propagators to vertices and fix their coupling constant evolution. These considerations encourage the hopes that the proposed program might work.

6.5 Hopf algebra formulation of unitarity and failure of perturbative unitarity in TGD framework

Unitarity is the basic property of the S-matrix and this raises the question whether also unitarity might be expressed using Hopf algebra language. Since perturbative unitarity giving infinite number of conditions in various orders of coupling constant is the basic aspect of quantum field theories, one is led to ask whether the generalized Feynmann diagrams could allow to identify the counterpart of perturbative unitarity. It turns out that this is not the case and is expected to be so on basis of quantum criticality.

That perturbative unitarity allows Hopf algebra formulation has been demonstrated to be the case by Yong Zhang [26] in the case of Φ^4 theory and the proof seems to generalize in a straightforward manner. The Hopf algebra formulation for unitarity generalizes at the formal level to TGD framework although the perturbative unitarity is lost.

6.5.1 Cutkosky rules

Scalar field propagator Δ_F can be decomposed into a sum of positive and negative cutting propagators Δ_+ and Δ_-.

$$\begin{aligned}
\Delta_F &= \theta(x^0-y^0)\Delta_+(x-y)+\theta(y^0-x^0)\Delta_-(x-y) \ , \\
\Delta_\pm(x-y) &= \int \frac{d^4k}{(2\pi)^4}\theta(\pm k^0)2\pi\delta(k^2-m^2)exp(ik\cdot(x-y)) \ . \quad (73)
\end{aligned}$$

Perturbative unitarity can be formulated using Cutkosky rules [27] in the following manner.

1. Consider the decompositions of the vertices of the Feynman diagram to ordinary and circled vertices by a cut line dividing the diagram to left and right sub-diagrams. Perform Hermitian conjugation for the circled vertices and for the propagators connecting circled vertices. If the propagator connects ordinary (circled) vertex to circled (ordinary) vertex, replace it by Δ_+ (Δ_-). This operation obviously puts the particles at the cut lines on mass shell. Conservation laws give constraints on admissible cuts.

2. Perturbative unitarity states that the sum of Feynman diagrams obtained by dividing the Feynman diagram Γ in all possible manners to ordinary and circled vertices using cut line and performing some modifications vanishes. Also included are the decomposition to the union of Γ and empty diagram Φ and to the union of Φ and conjugate of Γ: this obviously corresponds to the imaginary part $i(T-T^\dagger)$ of the scattering amplitude whereas the remaining terms correspond to the $TT^\dagger$ term in a given order of the perturbation theory defined by the number of vertices in the case of Φ^4 theory.

6.5.2 Largest time equation Hopf algebraically

Consider next the Hopf algebra formulation of the largest time equation (not equivalent with perturbative unitarity).

1. The set H defining the Hopf algebra is generated by connected Feynman diagrams. The number field is C. Algebra structure is obtained in a trivial manner. The addition of diagrams is defined by the formal linear combination $a\Gamma_1 + b\Gamma_2$. The multiplication m corresponds to disjoint union. The unit map η specifies unit e as empty set Φ with $\eta(1) = e$.

2. To define co-algebra structure some notational conventions are necessary. Let Γ denote arbitrary Feynman graph, $\mathcal{V}_N(\Gamma)$ denote the set of vertices of N-vertex Feynman diagram, γ a sub-diagram constructed using a subset $\mathcal{V}_c(\Gamma)$ of vertices of Γ and all lines connecting them. A reduced diagram Γ/γ is what remains when γ is cut out of Γ. The cut internal lines of Γ appear as external lines of γ and Γ/γ. Γ and empty set Φ are called trivial diagrams.

3. The co-product Δ is defined by

$$\Delta(\Gamma) = e \otimes \Gamma + \Gamma \otimes e + \sum_{1 \leq c < N} \gamma(\mathcal{V}_c) \otimes \Gamma/\gamma \ . \qquad (74)$$

   Here the sum includes all possible divisions of the vertices to ordinary and circled vertices.

The co-unit ϵ vanishes except for $\epsilon(e) = 1$. Co-associativity is ensured by the fact that two subsequent divisions of a vertex set commute. Antipode is defined recursively by

$$S(\Gamma) = -\Gamma - e \otimes \Gamma + \Gamma \otimes e + \sum_{1 \leq c < N} S(\gamma(\mathcal{V}_c)) \otimes \Gamma/\gamma \qquad (75)$$

with $S(e) = e$.

Define a new multiplication m of diagram and conjugate diagram by reconnecting the cut lines ending at the same point and assigning to the resulting internal line Δ_+. Let ϕ *resp.* ϕ_c be the Hopf algebra homomorphisms assigning to the Feynman diagram Feynman integral *resp.* its conjugate. m is in a well-defined sense the reverse of Δ and hence very natural.

The convolution $\phi \star \phi_c$ can be defined using the general formula involving only Δ and m

$$\phi \star \phi_c(\Gamma) \quad = \quad m(\phi \otimes \phi_c)\Delta(\Gamma) \ . \tag{76}$$

Largest time equation corresponds to the condition

$$\phi \star \phi_c(\Gamma) \quad = \quad \phi_c \star \phi(\Gamma) = 0 \tag{77}$$

6.5.3 The formulation of perturbative unitarity using cutting equation

Cutting equation stating perturbative unitary is obtained when only so called admissible cuts are allowed in the definition of the co-product Δ. An admissible cut is defined by a line dividing the diagram into two parts, the left and the right part, in a manner consistent with conservation laws. The left part γ connects at least one incoming line whereas the right part Γ/γ connects at least one outgoing line. For γ *resp.* Γ/γ ordinary *resp.* conjugate Feynman rules are applied.

Cutting equations are of the same Hopf-algebraic form as the largest time equation with the only difference coming from the different definition of the co-product:

$$\phi \star \phi_c(\Gamma) \quad \equiv \quad m(\phi \otimes \phi_c)\Delta(\Gamma) = 0 \ . \tag{78}$$

The convolution operation $\star$ is defined by the general formula with m defined by reconnecting the cut lines with the same quantum numbers so that Δ_- connects a circled vertex to an un-circled one. A sum over connected diagrams expressing unitarity condition in a given order of coupling constant defined by the number of vertices results in the operation. By writing explicitly the formula one has

$$\phi_c(\Gamma) + \phi(\Gamma) + \sum'' m\left[\phi(\gamma) \otimes \phi_c(\Gamma/\gamma)\right] \quad = \quad 0 \ . \tag{79}$$

Here the super-script $''$ denotes sum over all admissible cuttings. The first terms obviously correspond to $i(T - T^\dagger)$ term and latter terms to the sum of intermediate states appearing in $TT^\dagger$ term.

6.5.4 Hopf algebra formulation of unitarity conditions in TGD context

The equivalence of the generalized tree diagrams with an infinite number of generalized loop diagrams implied by quantum criticality can be regarded as something highly non-perturbative, so that the generalization of the perturbative unitarity is not expected to make sense.

This indeed seems to be the case. If perturbative unitarity would make sense, each loopy version of a given tree diagram would define a particular perturbative unitarity condition. The absorbtive part $i(T - T^\dagger)$ of a given tree amplitude would have infinitely many different expressions of form $TT^\dagger$ with the product of T and $T^\dagger$ defined such that only a sum over a subspace of intermediate states occurs in it. Unless the different subspaces of intermediate states are gauge equivalent, unitarity requires that absorbtive part equals to the sum over all these subsets.

The Hopf algebra formulation for unitarity could however makes sense also now.

1. There are good hopes that the stringy propagators G, G_+, and G_- are well defined. Since four-momentum appears as an argument of the propagator G, G_+ can be obtained as the absorptive part of G behaving as $G \propto 1/(p^2 - m^2 + i\epsilon)$ near mass shell and becomes thus restricted on mass shell. G_- can be defined as a Hermitian conjugate of G_+.

2. The co-product Δ for the tree diagram Γ must be defined as a sum over all possible loopy diagrams equivalent with Γ by assigning to each loopy diagram a sum over all admissible cuttings. Antipode S is defined recursively and summing over all loopy diagrams equivalent with Γ. m is essentially the inverse of Δ and defined in the same manner as in quantum field theory for each sub-diagram in the unitarity sum defined by $TT^\dagger$. Also the convolution $\star$ would be defined as before. Unitarity conditions formulated as cutting rules would state the vanishing of $\phi \star \phi_c(\Gamma)$.

7 The spectrum of the zeros of Riemann Zeta and physics

The properties of the spectrum of the zeros of Riemann Zeta have crucial implications for the quantum TGD. For instance, the question whether the imaginary parts of the zeros are linearly independent or not has deep physical meaning as the model for the scalar field propagator as a partition function for the super-canonical algebra demonstrates. Combining the existing intriguing numerical findings about the correlation functions of the non-trivial zeros allows to make a precise guess about the structure of the zeros of Zeta satisfying also the most obvious physical constraints.

7.1 Are the imaginary parts of the zeros of Zeta linearly independent of not?

Concerning the structure of the weight space of super-canonical algebra the crucial question is whether the imaginary parts of the zeros of Zeta are linearly independent or not. If they are independent, the space of conformal weights is infinite-dimensional lattice. Otherwise points of this lattice must be identified. The model of the scalar propagator identified as a suitable partition function in the super-canonical algebra for which the generators have zeros of Riemann Zeta as conformal weights demonstrates that the assumption of linear independence leads to physically unrealistic results and the propagator does not exist mathematically for the entire super-canonical algebra. Also the findings about the distribution of zeros of Zeta favor a hypothesis about the structure of zeros implying a linear dependence.

7.1.1 Imaginary parts of non-trivial zeros as additive counterparts of primes?

The natural looking (and probably wrong) working hypothesis is that the imaginary parts y_i of the nontrivial zeros $z_i = 1/2 + y_i$, $y_i > 0$, of Riemann Zeta are linearly independent. This would mean that y_i define play the role of primes but with respect to addition instead of multiplication. If there exists no relationship of form $y_i = n2\pi + y_j$, the exponents e^{iy_i} define a multiplicative representation of the additive group, and these factors satisfy the defining condition for primeness in the conventional sense. The inverses e^{-iy_i} are analogous to the inverses of ordinary primes, and the products of the phases are analogous to rational numbers.

There would exist an algebra homomorphism from $\{y_i\}$ to ordinary primes ordered in the obvious manner and defined as the map as $y_i \leftrightarrow p_i$. The beauty of this identification would be that the hierarchies of p-adic cutoffs identifiable in terms of the p-adic length scale hierarchy and y-cutoffs identifiable in terms p-adic phase resolution (the higher the p-adic phase resolution, the higher-dimensional extension of p-adic numbers is needed) would be closely related. The identification would allow to see Riemann Zeta as a function relating two kinds of primes to each other.

A rather general assumption is that the phases p^{iy_i} are expressible as products of roots of unity and Pythagorean phases:

$$\begin{aligned} p^{iy} &= e^{i\phi_P(p,y)} \times e^{i\phi(p,y)} \ , \\ e^{i\phi_P(p,y)} &= \frac{r^2 - s^2 + i2rs}{r^2+s^2} \ , \ r = r(p,y) \ , \ s = s(p,y) \ , \\ e^{i\phi(p,y)} &= e^{i\frac{2\pi m}{n}} \ , \ m = m(p,y) \ , \ n = n(p,y) \ . \end{aligned} \tag{80}$$

If the Pythagorean phases associated with two different zeros of zeta are different a linear independence over integers follows as a consequence.

Pythagorean phases form a multiplicative group having "prime" phases, which are in one-one correspondence with the squares of Gaussian primes, as its generators and Gaussian primes which are in many-to-one correspondence with primes $p_1 \ mod \ 4 = 1$. If p^{iy} is a product of algebraic phase and Pythagorean phase for any prime p, one should be able to decompose any zero y into two parts $y = y_1(p) + y_P(p)$ such that one has

$$log(p)y_1(p) = \frac{m2\pi}{n} \ , \ log(p)y_P(p) = \Phi_P = arctan\left[\frac{2rs}{r^2+s^2}\right] \ . \tag{81}$$

Note that the decomposition is not unique without additional conditions. The integers appearing in the formula of course depend on p.

7.1.2 Does the space of zeros factorize to a direct sum of multiples Pythagorean prime phase angles and algebraic phase angles?

As already noticed, the linear independence of the y_i follows if the Pythagorean prime phases associated with different zeros are different. The reverse of this implication holds also true. Suppose that there are two zeros $log(p)y_{1i} = \Phi_{P_1} + q_{1i}2\pi$, $i = a,b$ and two zeros $log(p)y_{2i} = \Phi_{P_2} + q_{2i}2\pi$, $i = a,b$, where q_{ij} are rational numbers. Then the linear combinations $n_1y_{1a} + n_2y_{2a}$ and $n_1y_{1b} + n_2y_{2b}$ represent same zeros if one has $n_1/n_2 = (q_{2a} - q_{2b})/(q_{1b} - q_{1a})$.

One can of course consider the possibility that linear independence holds true only in the weaker sense that one cannot express any zero of zeta as a linear combination of other zeros. For instance, this guarantees that the super-canonical algebra generated by generators labelled by the zeros has indeed these generates as a minimal set of generating elements.

For instance, one can imagine the possibility that for any prime p a given Pythagorean phase angle $log(p)y_{P_k}$ corresponds to a set of zeros by adding to $\Phi_{P_k} = log(p)y_{P_k}$ rational multiples $q_{k,i}2\pi$ of 2π, where $Q_p(k) = \{q_{k,i}|i = 1, 2, ..\}$ is a subset of rationals so that one obtains subset $\{\Phi_{P_k} + q_{k,i}2\pi|q_{k,i} \in Q_p(k)\}$. Note that the definition of y_P involves an integer multiple of 2π which must be chosen judiciously: for instance, if y_P is taken to be minimal possible (that is in the range $(0, \pi/2)$, one obviously ends up with a contradiction. The same is true if $q_{k,i} < 1$ is assumed. Needless to say, the existence of this kind of decomposition for every prime p is extremely strong number theoretic condition.

The facts that Pythagorean phases are linearly independent and not expressible as a rational multiple of 2π imply that no zero is expressible as a linear combination of other zeros whereas the linear independence fails in a more general sense as already found. An especially interesting situation results if the set $Q_p(k)$ for given p does not depend on the Pythagorean phase so that one can write $Q_p(k) = Q_p$. In this case the set of zeros of Zeta would be obtained as a union of translates of the set Q_p by a subset of Pythagorean phase angles and approximate translational invariance realized in a statistical sense would result. Note that the Pythagorean phases need not correspond to Pythagorean prime phases: what is needed is that a multiple of the same prime phase appears only once.

An attractive interpretation for the existence of this decomposition to Pythagorean and algebraic phases factors for every prime is in terms of the p-adic length scale evolution. The possibility to express the zeros of Zeta in an infinite number of manners labelled by primes could be seen as a number theoretic realization of the renormalization group symmetry of quantum field theories. Primes p define kind of length scale resolution and in each length scale resolution the decomposition of the phases makes sense. This assumption implies the following relationship between the phases associated with y:

$$\frac{\left[\Phi_{P(p_1)} + q(p_1)2\pi\right]}{log(p_1)} = \frac{\left[\Phi_{P(p_2)} + q(p_2)2\pi\right]}{log(p_2)} . \tag{82}$$

In accordance with earlier number theoretical speculations, assume that $log(p_2)/log(p_1) \equiv Q(p_2, p_1)$ is rational. This condition allows to deduce how the phases p_1^{iy} transform in $p_1 \rightarrow p_2$ transformation. Let $p_1^{iy} = U_{P,p_1,y}U_{q,p_1,y}$ be the representation of p_1^{iy} as a product of Pythagorean and algebraic phases. Using the previous equation, one can write

$$p_2^{iy} = U_{P,p_2,y}U_{q,p_2,y} = U_{P,p_1,y}^{Q(p_2,p_1)}U_{q,p_1,y}^{Q(p_2,p_1)} \quad . \tag{83}$$

This means that the phases are mapped to rational powers of phases. In the case of Pythagorean phases this means that Pythagorean phase becomes a product of some Pythagorean and an algebraic phase whereas algebraic phases are mapped to algebraic phases. The requirement that the set of phases p_2^{iy} is same as the set of phases p_1^{iy} implies that the rational power $U_{P,p_1,y}^{Q(p_2,p_1)}$ is proportional to some Pythagorean phase U_{P,p_1,y_1} times algebraic phase U_q such that the product of $U_qU_{q,p_1,y}^{Q(p_2,p_1)}$ gives an allowed algebraic phase. The map $U_{P,p_1,y} \rightarrow U_{P,p_1,y_1}$ from Pythagorean phases to Pythagorean phases induced in this manner must be one-to one must be the map between algebraic phases. Thus it seems that in principle the hypothesis might make sense.

The basic question is why the phases q^{iy} should exist p-adically in some finite-dimensional extension of R_p for every p. Obviously some function coding for the zeros of Zeta should exist p-adically. The factors $G_q = 1/(1-q^{-iy-1/2})$ of the product representation of Zeta obviously exist if this assumption is made for every prime p but the product is not expected to converge p-adically.

Also the logarithmic derivative of Zeta codes for the zeros and can be written as

$$\frac{\zeta'}{\zeta} = -\sum_q log(q)\frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{84}$$

As such this function does not exist p-adically but dividing by $log(p)$ one obtains

$$\frac{1}{log(p)}\frac{\zeta'}{\zeta} = -\sum_q Q(q,p)\frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{85}$$

This function exists if the p-adic norms rational numbers $Q(q,p)$ approach to zero for $q \rightarrow \infty$: $|Q(q,p)|_p \rightarrow 0$ for $q \rightarrow \infty$. The p-adic existence of the logarithmic derivative would thus give hopes of universal coding for the zeros of Zeta and also give strong constraints to the behavior of the factors $Q(q,p)$. The simplest guess would be $Q(q,p) \propto p^q$ for $q \rightarrow \infty$.

7.1.3 Correlation functions for the spectrum of zeros favor the factorization of the space of zeros

The idea that the imaginary parts of the zeros of Zeta are linearly independent is a very attractive but must be tested against what is known about the distribution of the zeros of Zeta.

There exists numerical evidence for the linear independence of y_i as well as for the hypothesis that the zeros correspond to a union of translates of a basic set Q_1 by subset of Pythagorean phase angles. Lu and Sridhar have studied the correlation among the zeros of ζ [38]. They consider the correlation functions for the fluctuating part of the spectral function of zeros smoothed out from a sum of delta functions to a sum of Lorentzian peaks. The correlation function between two zeros with a constant distance $K_2 - K_1 + s$ with the first zero in the interval $[K_1, K_1 + \Delta]$ and second zero in the interval $[K_2, K_2 + \Delta]$ is studied. The choice $K_1 = K_2$ assigns a correlation function for single interval at K_1 as a function of distance s between the zeros.

1. The first interesting finding, made already by Berry and Keating, is that the peaks for the negative values of the correlation function correspond to the lowest zeros of Riemann Zeta (only those contained in the interval Δ can appear as minima of correlation function). This phenomenon observed already by Berry and Keating is known as resurgence. That the anti-correlation is maximal when the distance of two zeros corresponds to a low lying zero of zeta can be understood if linear combinations of the zeros of Zeta are the least probable candidates for zeros. Stating it differently, large zeros tend to avoid the points which represent linear combinations of the smaller zeros.

2. Direct numerical support the hypothesis that the correlation function is approximately translationally invariant, which means that it depends on $K_2 - K_1 + s$ only. Correlation function is also independent of the width of the spectral window Δ. In the special $K_1 = K_2$ the finding means that correlation function does not depend at all on the position K_1 of the window and depends only on the variable s. Prophecy means that the correlation function between the interval $[K, K + \Delta]$ and its mirror image $[-K - \Delta, -K]$ is the correlation function for the interval $[2K + \Delta]$ and depends only on the variable $2K + s$ allowing to allows to deduce information about the distribution of zeros outside the range $[-K, K]$. This property obviously follows from the proposed

hypothesis implying that the spectral function is a sum of translates of a basic distribution by a subset of Pythagorean prime phase angles.

This hypothesis is consistent with the properties of the smoothed out spectral density for the zeros given by

$$\langle \rho(k) \rangle = \frac{1}{2\pi} log(\frac{k}{2\pi}) \ . \tag{86}$$

This implies that the smoothed out number of zeros y smaller than Y is given by

$$N(Y) = \frac{Y}{2\pi}(log(\frac{Y}{2\pi}) - 1) \ . \tag{87}$$

$N(Y)$ increases faster than linearly, which is consistent with the assumption that the distribution of zeros with positive imaginary part is sum over translates of a single spectral function ρ_{Q_0} for the rational multiples $q_i X_p$, $X_p = 2\pi/log(p)$, $q_i \in Q_p$, for every prime p.

If the smoothed out spectral function for $q_i \in Q_p$ is constant:

$$\rho_{Q_p} = \frac{1}{K_p 2\pi} \ , \ K_p > 0 \ , \tag{88}$$

the number $N_P(Y,p)$ of Pythagorean prime phases increases as

$$N_P(Y|p) = K_p(log(\frac{Y}{2\pi}) - 1) \ , \tag{89}$$

so that the smoothed out spectral function associated with $N_P(Y|p)$ is given by the function

$$\rho_P(k|p) = \frac{K_p}{k} \tag{90}$$

for sufficiently large values of k. Therefore the distances between subsequent zeros could quite well correspond to the same Pythagorean phase for a given p and thus should allow to deduce information about the spectral function ρ_{Q_0}. A convenient parametrization of K_p is as $K = K_{p,0}/4\pi^2$ since the points of Q_p are of form $q_i 2\pi = (n(q_i) + q_1(q_i))2\pi$, $q_1 < 1$, and $n(q_i)$ must in the average sense form an evenly spaced subset of reals.

7.1.4 Physical considerations favor the linear dependence of the zeros

The numerical evidence is at best suggestive and one can always argue that by an arbitrary small deformation of the linearly dependent zeros one obtains linearly independent zeros. This would however require that each zero of form $y_{P_i} + q2\pi$, $q \in Q_p$ is very near to a zero $\Phi_{P_{k(i,q)}} + q_{k(i,q)}2\pi$. In other words, the union of the translates of Q_p by a subset of Pythagorean phases would approximate the zeros in one-one correspondence with a larger subset of Pythagorean phases (given prime phase appears only once). This should hold for every prime and this seems rather implausible.

On the other hand, the linear dependence between zeros has deep physical implications for the basic quantum TGD, and as the following arguments demonstrate, is physically highly desirable. The precise arguments are developed later and here only the skeleton of the argument is given.

1. The zeros label the generating elements of the super-canonical algebra and the failure of the linear independence means that the weight system is not just the infinite-dimensional lattice spanned by the zeros but can be regarded as a kind of bundle like structure such that the linear combinations $log(p)y_b = \sum_{i=1}^{N} n_i \Phi_{P_{k_i}}$ form N-dimensional lattice and the fiber at a given point of this lattice consists of the points $log(p)y_f = \sum_i n_i q_i 2\pi$. The set of these points is the lattice $n_1 Q_p \times n_2 Q_p \times ...$ divided by the equivalence defined by $y_{f,1} = y_{f,2}$ and for given values of n_i a discrete analog of the one-dimensional space of parallel hyper-planes of an N-dimensional defined by the equation $\sum_{i=1}^{N} n_i x^i = y$ space parameterized by the values of y. What is essential that the space of the planes is different for each point $y_b = \sum_{i=1}^{N} n_i y_{P_{k_i}}$.

2. The calculation of the scalar propagator as a partition function for the super-canonical algebra assuming linear independence gives without any restrictions to the super-canonical weights an infinite number of delta-function resonances of form $\delta(p^2 - m_n^2)$, and at the limit when all zeros of the Riemann Zeta are included in the sub-algebra of super-canonical algebra the set of delta function resonances defines a dense set on real axis. If only the super-canonical conformal weights generated by the positive zeros of Zeta are included, delta function resonances become ordinary poles of form $1/(p^2 - m_k^2)$. The resonances are infinitely narrow and form also now a dense set of real axis.

3. This result, which can be claimed to be non-physical, can be avoided if the zeros are not linearly independent. Although the partition function cannot be calculated explicitly in this case, one can expect that the linear independence gives a reasonable first approximation and that the failure of the approximation is due to the multiple counting caused by the neglect of the fact that the planes of the fiber space can contain several equivalent points. If the zeros are linearly dependent, resonances get a finite width and singularities are avoided for real values of the masses and there are good hopes that the partition function is well-defined for the entire super-canonical algebra.

4. A further argument favoring the proposed form of zeros relates to the two hierarchies strongly suggested by quantum TGD. The first hierarchy corresponds to ordinary primes labelling p-adic length scales and corresponds to length scale resolution. The second hierarchy corresponds to a hierarchy of algebraic extensions of p-adic numbers and there is strong feeling that this hierarchy should correspond to the hierarchy of Beraha numbers $B_n = 4cos^2(\pi/n)$ associated with the phases $q = exp(i\pi/n)$. The phases $exp(i\pi/p)$ or their non-trivial powers, for p prime, are even more interesting because of the structure of finite field $G(p,1)$. The phases q which are expressible using only rationals and their square roots and thus correspond to n-polygons constructible using only ruler and compass, are also interesting and seem to have concrete physical interpretation [C7, D5]. The expression for n is in this case $n = n_F = 2^k \prod_i F_{n_i}$, where $F_n = 2^{2^n} + 1$, $n = 0,..,4$ are Fermat primes. All Fermat primes in the product are different.

   One could consider the possibility that the rationals $q \in Q_p$ for any p can be ordered by their size in such a manner that this ordering corresponds to the ordering of primes with respect to size. Obviously the condition $Q_p = Q_1$ must hold true. This would imply that the products of the powers of the phases $exp(iq2\pi)$ for the lowest N values of q_i would give the Beraha phases corresponding to square free integers having corresponding primes p_i, $i = 1, ..., N$, as factors. All Beraha phases are obtained if the phases $exp(i\pi/p^n)$, $n = 1, 2, ..$ or their non-trivial powers, are also present. If this waves the case the full p-adic length scale hierarchy with powers of p would correspond to the hierarchy of Beraha phases. This would mean that the addition of new super-canonical conformal weights of increasing size to the sub-algebra of the super-canonical algebra would mean the increase of the dimension of the extension of p-adic numbers needed to represent the resulting phases p-adically as well as an increasing phase resolution.

5. With the assumptions about the structure of zeros of Zeta, the hierarchies defined by the subset y_{P_i} of multiples of Pythagorean prime phase angles and algebraic phases would neatly factorize and the latter would correspond to the p-adic length scale hierarchy. Pythagorean phases correspond to phases of the squares of Gaussian integers $r+is$ and the squares of Gaussian primes define naturally Pythagorean primes. The norm squared of the Gaussian prime is obviously prime: $r^2+s^2=p_1$, and satisfies $p_1 \ mod\ 4=1$. Hence there is a natural correspondence between Pythagorean prime phases and primes $p\ mod\ 4=1$. One can wonder whether also Pythagorean prime phase angles could be mapped to a subset of primes such that that size ordering for y_{P_i} would correspond to the size ordering for the subset of primes. As already noticed, the primeness property is actually an un-necessary strong requirement for Pythagorean phases: it is enough that only single power of a given Pythagorean prime phase appears.

Needless to emphasize, these speculative assumptions which could make possible to realize the p-adicization program and understand that origin of also effective p-adicity would pose very strong constraints on the spectrum of zeros and are certainly testable numerically.

7.1.5 The notion of dual Zeta

These considerations lead to the idea that Riemann Zeta has a dual for which the role of multiplicative primes is taken by the additive primes. This function, call it $\zeta_d(u)$ should either vanish or diverge at points $u=p$. The partition functions for super-canonical conformal weights define analogs of Riemann Zeta involving analog of restriction of summation to integers which are products of even and odd integers and these functions indeed are singular at powers $u=p^{kx}$, $x=2\pi k/y$, $k=1,2,...$, where the transcendental values x do not depend on p. That the singularities do not occur for rational values of u is physically very satisfactory since this would mean that the scattering rates could become infinite.

The precise dual ζ_d of ζ would be the function

$$\zeta_d(u) \quad = \sum_{\sum n(y)y,y\in Y} u^{i\sum n(y)y} = \prod_{y>0,y\in Y} \frac{1}{1-u^{iy}} \ , \tag{91}$$

where the summation is over all possible formal linear combinations of positive imaginary parts y of zeros or subset of them with non-negative coefficients $n(y)$. In the case that the zeros of Riemann Zeta are linearly independent, the set Y corresponds to all zeros. If the zeros are of the form $y = y_{P_i} + q2\pi$, $q \in Q_0$, one can restrict the consideration to a subset Y of zeros obtains by selecting only single value of $q \in Q_0$ for each y_{P_i}. The simplest option is that q is same for all values of y_{P_i}.

The interpretation as a product of bosonic partition functions defined by the zeros of ζ or subset of them, obviously makes sense, and the form of the partition function is the same as that of Riemann Zeta in the product representation. By writing $u = \rho exp(i\phi)$, $\phi \geq 0$ one finds that all terms in the product converge if the term corresponding to the smallest value $y_{min} \simeq 14.124725$ of y converges. This gives the condition $\phi > 1/y_{min} \sim 2\pi/14$. One can however extract arbitrary number of the lowest terms in the product as a separate well-defined factor and obtain a convergence above arbitrarily small $\phi_{min} = \epsilon > 0$. Thus the product is well-defined arbitrary near to real axis above it.

The limit $\phi \rightarrow 2\pi$ is well-defined and at $z = \rho e^{i2\pi}$, $\rho > 0$ the product can be written as

$$\zeta_d(\rho e^{i2\pi}) = \prod_{y \in Y} \frac{1}{1-\rho^{-2\pi y}\rho^{iy}} . \quad (92)$$

This expression converges to a finite result at the real axis and pole is not possible. This expression is not consistent with the requirement that $u \rightarrow 1/u$ induces a complex conjugation of ζ_d at the real axis.

The conjecture is that the limit $\phi \rightarrow 0_+$ limit of ζ_d vanishes or diverges for $u = p^{\pm 1}$. Also now the powers of $u_p = p^{kx}$ define poles of the individual factors in the product at real axis. For $u = p$ one can write

$$\zeta_d(p)\overline{\zeta}_d(p) = \prod_{y>0, y \in Y} \frac{1}{4sin^2\left[\frac{\phi(p,y)+\phi_P(y)}{2}\right]} . \quad (93)$$

Here U refers to the subset of zeros of Zeta. This expansion diverges for $sin^2[(\phi(p,y) + \phi_P(y))/2] < 1/4$ for sufficiently many values of y. An interesting possibility inspired by the connection with braid groups and Beraha numbers $B_n = 4cos^2(\pi/n)$ is that the numbers $4cos^2[\phi(p,y)]$ are Beraha numbers so that one would have $\phi(p,y) = \pi/n(p,y)$, $n(p,y) \geq 3$. For $n(p,y) \geq 3$ and $\phi_P(y) = 0$, all factors in the product would be larger than or equal to one so that the product would diverge. The vanishing would be thus due the Pythagorean phases. Of course, these arguments cannot be however taken completely seriously since the product expansion does not converge at the real axis.

Also the zeros $z_i = 1/2 + iy_i$, $y_i > 0$, are generators of an Abelian algebra with integers $n/2 + \sum_i n_i y_i$, $\sum n_i = n > 0$. The corresponding zeta function is

$$\zeta_d(u) = \prod_{y \in Y} \frac{1}{1 - u^{-\frac{1}{2} - iy}} \ . \tag{94}$$

This function has even nearer resemblance to the ordinary ζ. Interestingly, the product $\prod_d \zeta_d(p)$ satisfies the identity

$$\prod_p \zeta_d(p) = \prod_{y \in Y} \zeta(\frac{1}{2} + y) \ , \tag{95}$$

if one exchanges freely the order of producting. The fact that all factors on the right hand side vanish would suggest that also $\zeta_d(p)$ vanishes for all values of p.

7.2 Why the zeros of Zeta should correspond to number theoretically allowed values of conformal weights?

The following argument provides support for the belief that the conformal weights $s = 1/2 + iy$ for which $p^{1/2+iy}$ exist in a finite-dimensional extension of rationals for all values of prime p, indeed correspond to the non-trivial zeros of Zeta.

1. The basic idea of the number theoretical approach is that the conformal weights $1/2 + iy$ are such that the radial waves $r^{-1/2-iy}$ exist for all rational (and thus for integer) values of r in some finite-dimensional extension of rationals. The logarithms $log(n)$ of integers can be interpreted as quantum numbers of a system defined by an arithmetic quantum field theory and Zeta function $\zeta = \sum_n n^{-iy-1/2}$ with $s = 1/2 + iy$ interpreted as an inverse temperature, defines the partition function of this system.

2. On the other hand, so called Selberg's Zeta function characterizes the eigen values of the Laplacian in 2-dimensional quantum billiard systems defined in the fundamental domain of some hyperbolic subgroup

G of $SL(2,Z)$ acting in the hyperbolic plane $SL(2,R)/SO(2)$ [39]. The fundamental domain is analogous to a box containing the particle. At quantum level the boundary conditions are satisfied by summing over all the G translates of $SL(2,R)$ invariant Green function with respect to the second argument. Physically this is analogous to putting to all copies of the fundamental domain an image charge. The confinement to the fundamental domain selects from the continuous energy spectrum a discrete sub-spectrum. Selberg's Zeta (its logarithmic derivative) has the allowed energy eigen values as its zeros (poles). Furthermore, the energy eigen values of Laplacian are of form $E = -l(l+1)$, where $l = -1/2 - iy$ is identifiable as the counterpart of conformal weight and has the same form as the zeros of Zeta. y has discrete spectrum of values characterized by the choice of G. The density of the energy eigenvalues is amazingly similar to that of Zeta.

3. On basis of above resemblances one can argue that Riemann Zeta (its logarithmic derivative) characterizes the purely number theoretical spectrum as its zeros (poles). If this is the case, the zeros of Zeta would coincide with the number theoretically allowed conformal weights $1/2+iy$.

7.2.1 The p-adically existing conformal weights are zeros of Zeta for 1-dimensional systems allowing discrete scaling invariance

The obvious question is whether one could reduce number theory to symmetry. The following considerations suggests that $D \geq 2$-dimensional spaces do not allow a system having zeros of Zeta as its spectrum.

1. The density of states of the Selberg Zeta function differs in some aspects from that of Zeta so that Riemann Zeta probably has no interpretation as a Selberg Zeta function of a number theoretical system. For instance, the average density of states with respect to y grows linearly rather than logarithmically although the fluctuating part of the density of states is formally very similar to that of Zeta.

2. Lobatchevski space (the hyperboloid of the 4-dimensional future light cone) has $SL(2,C)$ as its isometry group. The energy spectrum of Laplacian in this case is of the form $E = -l(l+2) = 1 + y^2$ with $l = -1 - iy$ and thus different from the spectrum of 2-dimensional case and of Riemann Zeta. Due to the higher dimension of the system

the mean density of states grows even faster than in the 2-dimensional case so that there seems to be no hope of getting the density of states of Riemann Zeta.

Only one-dimensional systems give hopes of the required logarithmically varying mean density of states. The simplest candidate one can imagine is a system with discrete scaling invariance.

1. Instead of Laplacian, and in complete accordance with the view that conformal invariance is the key to the understanding of Riemann Zeta, one can consider the scaling operator $L_0 = xd/dx$ acting at the half line R_+ so that the Green functions defined by the equation

$$(L_0 + z)G(x, x_1) = (xd/dx + z)G(x, x_1) = \delta(\frac{x}{x_1} - 1) \tag{96}$$

   become the object of interest. The solution can be written as

$$G(x, x_1|z) \quad = \quad (\frac{x}{x_1})^z \times \theta(\frac{x}{x_1} - 1) \ . \tag{97}$$

   Here $\theta(x)$ denotes the step function. The requirement that the integrals

$$\int \overline{G}(x, x_1|z_1)G(x, x_1|z_2)dx$$

   reduce to the inner products of ordinary plane waves when $ln(x/y)$ is taken as an integration variable forces the condition $z = 1/2 + iy$. In fact, this might be seen as the physicist's "proof" of the Riemann hypothesis.

2. Following the construction of the automorphic Green functions in the hyperbolic plane described in [39], the next step is to form a sum over the $x-$ scaling transforms of $G(x, x_1|z)$ by summing over the integer scaled values nx of x to form a well defined Green function in the fundamental domain associated with the semigroup of integer scalings. Any interval $[n, 2n]$ forms a fundamental domain. This gives

$$\begin{aligned} G_I(x,x_1|\frac{1}{2}+iy) &= \sum_n G(nx,x_1|\frac{1}{2}+iy) = \sum_n (\frac{nx}{x_1})^{\frac{1}{2}+iy} \\ &= \zeta(\frac{1}{2}+iy)\times(\frac{x}{x_1})^{\frac{1}{2}+iy} \ . \end{aligned} \tag{98}$$

The resulting Green function is proportional to Riemann Zeta at the critical line and vanishes for the zeros of Zeta. Note that the logarithmic derivative of ζ divided by $log(p)$ exists in a finite-dimensional extension of R_p for $x = n/2 + i\sum_k m_k y_k$ if the basic number theoretical requirements on the phases p^{iy} defined by the zeros of Zeta are satisfied: in particular $log(p_1)/log(p)$ must have R_p norm which approaches zero for larger values of p_1. Hence the logarithmic derivative of Zeta could codes the number theoretical physics universally.

3. In the usual approach [39] the integral of G_I over the fundamental domain would give the density of states $d(E)$. In the recent case the integration over the fundamental domain $[1,2]$ gives just ζ function

$$\int_1^2 G_I(x,x|-\frac{1}{2}+iy)dx = \sum_n n^{-\frac{1}{2}-iy} = \zeta(\frac{1}{2}+iy) \ . \tag{99}$$

The interpretation as a density of states is obviously not possible. The proof for the Riemann hypothesis to be discussed later allows to interpret the vanishing of Riemann Zeta as as orthogonality of physical states labelled by zeros of Zeta with a tachyonic vacuum state with a vanishing conformal weight. The vanishing of Green function could also now have an interpretation stating that the physical states labelled by non-trivial zeros are orthogonal to the scaling invariant tachyonic vacuum.

4. Quite generally, the imaginary part of the logarithmic derivative of any real function $f(E)$ for which energy eigenvalues E_n correspond to zeros of unit multiplicity, defines the density of states as a sum over delta functions. $G(y) = \zeta(1/2+iy)$ is real at the critical line as is also its logarithmic derivative apart from delta function singularities of the imaginary part at the zeros of Zeta so that its logarithmic derivative indeed gives the density of zeros of Zeta:

$$d(y) \quad = \quad \frac{1}{\pi} Im \left[i \frac{dlog \left[\zeta(\frac{1}{2} + iy) \right]}{dy} \right] = \sum_n \delta(y - y_n) \ . \qquad (100)$$

This ultra simple model realizes the idea that the logarithmic derivative of Green function naturally associated with a system invariant under the semi-group of integer scalings codes as its poles the zeros of Zeta. The p-adic existence of the Green function in turn is equivalent with the requirement that the spectrum corresponds to the zeros of Zeta.

7.2.2 Realization of discrete scaling invariance as discrete 2-dimensional Lorentz invariance

Both the role of the hyperbolic groups and the fact that in quantum TGD zeros of Zeta label representations of Lorentz group, encourage to think that the 1-dimensional hyperbolic subspace $t^2 - x^2 = constant$ of 2-dimensional Minkowski space having Lorentz group $SO(1,1)$ as its symmetries realizes the above described system physically. The counterpart of the hyperbolic subgroup G of $SL(2,R)$ would the semigroup of Lorentz transformations defining integer scalings of the second light like coordinate:

$$u \equiv t + z \rightarrow nu \ , \quad v \equiv t - z \rightarrow \frac{1}{n} v \ .$$

This semigroup corresponds to the diagonal semi-subgroup of $SL(2,Q)$ consisting of matrices $diag(\lambda, 1/lambda) = diag(n, 1/n)$. The reduction to semigroup is natural by the presence of the p-adic length scale cutoff unavoidable in p-adicization.

Taking $u = t + z$ as the coordinate of the hyperboloid, the situation reduces to that already considered. Infinitesimal Lorentz boost acts as a scaling operator and its eigenvalues correspond to the zeros of Zeta by number theoretic existence requirements. The matrices $diag(p, 1/p)$, p prime, are completely analogous to the group elements g_0 defining primitive periodic orbits in the higher-dimensional case so that prime numbers are naturally realized as discrete Lorentz transformations. Prime Lorentz transformations and their inverses generate rational Lorentz group. The length of the primitive periodic orbit corresponds to the scaling parameter $log(p)$ defining the scaling by p as an exponentiated scaling transformation $u \rightarrow exp(log(p))u = pu$.

7.3 Zeros of Riemann Zeta as preferred super-canonical weights

Along standing heuristic basic hypothesis has been that the radial conformal weights Δ assignable to the functions $(r_M/r_0)^\Delta$ of the radial coordinate r_M of $\delta M^4_\pm$ in super-canonical algebra consisting of functions in $\delta M^4_\pm \times CP_2$ are expressible as linear combinations of zeros of Riemann Zeta. Quantum classical correspondence in turn inspires the hypothesis that these conformal weights can be mapped to the points of a geodesic sphere of CP_2 playing the role of conformal heavenly sphere.

The following arguments suggest that radial conformal weights in fact depend on the point of geodesic sphere $S^2 \subset CP_2$ and are given in terms of the inverse of Riemann Zeta having the natural complex coordinate of S^2 as argument. This implies a mapping of the radial conformal weights to the points of the geodesic sphere CP_2. Linear combinations of zeros correspond to algebraic points in the intersections of real and p-adic space-time sheets and are thus in a unique role from the point of view of p-adicization. This if one believes the basic conjecture that the numbers p^s, p prime and s zero of Riemann Zeta are algebraic numbers. The prediction is that the zeros of Riemann Zeta correspond to radial conformal weights for which system is critical against a phase transition changing the value of Planck constant [C1]. Zeros of Zeta have been indeed associated with critical systems. One application would be to high T_c superconductivity.

7.3.1 Basic argument

The basic argument runs as follows.

1. Let us start from the idea that the discrete set of points of the geodesic sphere S^2 of CP_2 labelling commuting R-matrices should correspond to the super-canonical conformal weights Δ assignable to the functions $(r_M/r_0)^\Delta$ of the radial light-like coordinate r_M of $\delta M^4_\pm$ in super-canonical Hamiltonians.

2. This discrete set of conformal weights should correspond to a discrete set of points at the partonic 2-surface X^2 defined as the intersection of 3-D light-like causal determinant X^3_l defining the orbit of parton and $\delta M^4_\pm \times CP_2$. This selection of a discrete subset of points in the CP_2 projection of X^2 would make it possible to realize radial conformal weights as points of a "heavenly sphere" defined by the CP_2 projection. A homologically non-trivial geodesic sphere $S^2 \subset CP_2$ would provide a natural complex coordinate for the projection with S^2 isometries

acting as Möbius transformations for the preferred complex coordinate z shared by X^2 and $S^2 \subset CP_2$. This assumption is however unnecessarily strong as will be found.

3. The finite set of points having interpretation as a braid would belong to a "time=constant" section of 2-dimensional "space-time", presumably circle, defining physical states of a two-dimensional conformal field theory for which the scaling operator L_0 takes the role of Hamiltonian.

7.3.2 Radial conformal weight as a function of CP_2 coordinates

One can criticize the idea of assigning radial conformal weights with CP_2 points as an ad hoc procedure. Nothing however prevents of modifying the original definition of Hamiltonians of $\delta M^4_\pm \times CP_2$.

1. Suppose that the factors $1/(1+p^{iy})$ of Riemann Zeta in the product representation are universal for its zeros in the sense that the power p^{iy} for primes p always belong to some finite-dimension extension of rationals for the zeros $s = 1/2 + iy$ of ζ.

2. One could *redefine* the configuration space Hamiltonians by assuming that radial conformal weights Δ are functions of CP_2 coordinates so that the radial parts of Hamiltonians would be of form $(r_M/r_0)^{\Delta(s)}$.

3. For instance, one could have $\Delta = \zeta^{-1}(\xi^1/\xi^2)$, where ξ^i are complex CP_2 coordinates transforming linearly under group $U(2)$: CP_2 itself parameterizes these coordinate choices. For a given value of $r = r_M/r_0$ one should select some branch of the inverse of ζ in this formula. Branches are in one-one correspondence with zeros of Zeta and they could be in one-one correspondence with partonic 2-surfaces assignable to a given 3-surface. If the number theoretical universality holds true then the values of $\xi^1/\xi^2 = \zeta(\sum_k n_k s_k)$, where s_k are zeros of ζ, define the preferred points of CP_2 and thus of partonic 2-surface and conformal weights are quantized number theoretically. An interesting question is whether these values of ζ are algebraic numbers.

4. This picture suggest a large number of fractal hierarchies with anomalous dimensions defined by the linear combinations of imaginary parts of zeros of Zeta. At a given partonic 2-surface the branch s_1 of $s = \zeta^{-1}$ can change to s_2 at radial points satisfying $(r_M/r_0)^{s_1} =$

$(r_M/r_0)^{s_2}$. This gives $Re(s_1) = Re(s_2)$ so that the change can occur only along a line parallel to the critical line. The second condition is $(Im(s_1) - Im(s_2))log(r_M/r_0) = n2\pi$ and can be satisfied for any pair of linear combinations of zeros of Zeta at points $r_M/r_0 = exp[Im(s_1) - Im(s_2))n2\pi]$.

5. One can worry about the non-uniqueness of the selection of preferred CP_2 coordinates. As a matter fact, in its recent form the construction of configuration space as a union of sub-configuration spaces involves, not only a selection of the tip of the light cone of M^4, but also a selection of a preferred point of CP_2 so that the points of H label sub-configuration spaces [C8]. The physical interpretation is in terms of a geometric correlate for the selection of Cartan algebra of commuting color charges.

6. The requirement that the Hamiltonian belongs to an algebraic extension of p-adic numbers forced by the p-adicization constraint would select a discrete set of points of X^2. Thus the discrete spectrum of conformal weights expressible in terms of zeros of Riemann Zeta would result from the number theoretical quantization forced by the p-adicization constraint.

7.3.3 Why a discrete set of points of partonic 2-surface must be selected?

As already noticed, p-adicization might provide a deeper motivation for the selection of discrete subset of points of partonic 2-surface in the construction of S-matrix elements in the case of non-diagonal transitions between different number fields.

1. The fusion of p-adic variants of TGD with real TGD, could be possible by algebraic continuation. This however requires the restriction of n-point functions to a finite set of algebraic points of X^2 with the usual stringy formula formula for S-matrix elements involving an integral over a circle of X^2 replaced with a sum over these points.

2. The same universal formula would give not only ordinary S-matrix elements but also those for p-adic-to-real transitions describing transformation intentions to actions. Quite generally, the formula would express S-matrix elements for transitions between two arbitrary number fields as algebraic numbers so that p-adicization of the theory would become trivial.

3. The interpretation of this finite set of points as a braid suggests a connection with the representation of Jones inclusions in terms of a hierarchy of braids [C7, E9] with the increasing number of strands meaning a continually improved finite-dimensional approximation of the hyper-finite factor of type II_1 identifiable as the Clifford algebra for the configuration space. The hierarchy of approximations for the hyper-finite factor would correspond to a genuine physical hierarchy of S-matrices corresponding to increasing dimension of algebraic extension of various p-adic numbers. This hierarchy would also define a cognitive hierarchy.

What could then be this discrete set of points having interpretation as a braid?

1. Number theoretical vision suggests that quantum TGD involves the sequence hyper-octonions $\rightarrow$ hyper-quaternions$\rightarrow$ complex numbers$\rightarrow$ reals $\rightarrow$ finite field $G(p,1)$ or of its algebraic extension. These reductions would define number theoretical counterparts of dimensional reductions. The points in the finite field $G(p,1)$ could be defined by p-adic integers modulo p so that a connection with p-adic numbers would emerge. Also more general algebraic extensions of p-adic numbers are allowed.

2. If the exponents p^{iy} for the zeros $z = 1/2 + iy$ of Zeta are algebraic numbers, the linear combinations for a finite subset of them for a given algebraic extension of p-adic numbers would naturally represent preferred points of $S^2 \subset CP_2$ using the group-theoretically preferred complex coordinate z of S^2. Note that radial coordinate which itself is expressible in terms of CP_2 coordinates when partonic 2-surface has 2-dimensional CP_2 projection, must be rational which poses also conditions on the allowed set of points.

3. If the radial conformal weights are expressible in terms of CP_2 coordinates, such that for a restriction to $S^2 \subset CP_2$ one has $\Delta = \zeta^{-1}(z = \xi^1/\xi^2)$, the basic condition would be that $r = r_M/r_0$ is rational and r^Δ belongs to the algebraic extension of p-adic numbers used. The rationality of $r_M(z)$ as function of coordinate z would select a subset of linear combinations $s = \sum n_k s_k$ of zeros of Zeta. This is not the only possibility. If one assumes that scaling invariant integration measure dr/r defines the inner product then $s = 1/2 + \sum_k n_k y_k$ defines a set of orthogonal "planewaves".

7.3.4 What is the fundamental braiding operation?

The basic quantum dynamics of TGD could define the braiding operation for the braid defined by a discrete set of points of X^2 satisfying the algebraicity conditions.

1. The complex coordinate z of X^2 is defined by the conformal equivalence class of the induced metric of X^2 apart from conformal transformation and positions of punctures are expressed using this coordinate.

2. The selection of the radial coordinate r_M defines a rest system and thus a foliation of $M^4_\pm$ by light-cone boundaries parameterized by M^4 time coordinate m^0 giving the temporal location of the tip of $\delta M^4_\pm$ along the line $r_M = 0$. This foliation defines also a foliation of the partonic orbit defined by the light-like 3-surface X^3_l by partonic 2-surfaces $X^2(m^0)$. Each of these surfaces defines a number theoretic braid and one expects that the positions of the punctures of braid expressed using coordinate z evolve most of the time in a continuous manner so that the braiding flow is well-defined. If one has $\Delta = \zeta^{-1}(z_1 = \xi^1/\xi^2)$ then z_1 remains constant during the flow and only z changes meaning that M^4 projection corresponding to z_1 changes. Hence, if M^4 projection is 2-dimensional, the braiding flow can be regarded as a flow defined at M^4 projection, which of course conforms with the physical picture.

3. More concretely, let M^4 projection of X^2 be 2-dimensional and CP_2 projection 1-dimensional. For a given linear combination $s = \sum n_k s_k$ of zeros of Riemann Zeta the inverse image is a 1-dimensional curve of X^2 and the rational values for the graph of r_M at this curve with the property that $(r_M/r_0)^s$ belongs to the algebraic extension of p-adic numbers in question define the threads of the braid. The flow means that the position of these punctures in coordinate plane defined by the complex coordinate z move. If CP_2 projection is 2-dimensional, the set of values of rational values of r_M for which $(r_M/r_0)^s$ belongs to the algebraic extension of p-adic numbers, consists of discrete points and by a judicious choice of r_0 this can be guaranteed always.

4. New points can appear to and old points disappear from the braid so that a structure more general than a mere braid is in question. It would not be surprising if the points satisfying algebraicity conditions disappear or are created in pairs. If this is the case, a tangle like structure allowing topological particle pair creation and annihilation would

be in question. Indeed, if the CP_2 projection of X^2 is 1-dimensional, the rational values of r_M/r_0 satisfying the basic constraint appear pairwise around extrema and when the shape of the graph of r_M changes, these values disappear or appear in pairwise manner.

5. This picture makes sense also for macroscopic 2-surfaces defining outer boundaries of physical systems (quantum Hall effect and topological quantum computation [E9]).

7.3.5 Do zeros of Riemann ζ correspond to critical conformal weights for $\hbar$ changing phase transitions?

Zeros of Riemann Zeta have been for long time speculated to closely relate to fractal and critical systems. If the proposed general ansatz for super-canonical radial conformal weights holds true, these speculations find a mathematical justification.

Geometrically the transition changing the value of $\hbar(M^4)$ correspond to a leakage of partonic 2-surfaces between different copies of $M^4 \times CP_2$ with same CP_2 factor and thus same value of $\hbar(CP_2)$. Critical 2-surfaces can be regarded as belonging to either factor which means that points of critical 2-surfaces must correspond to the CP_2 orbifold points, in particular, $z = \xi^1/\xi^2 = 0$ and $z = \xi^1/\xi^2 = \infty$ remaining invariant under the group $G \subset SU(2) \subset SU(3)$ defining the Jones inclusion, that is the north and south poles of homologically non-trivial geodesic sphere $S^2 \subset CP_2$ playing the role of heavenly sphere for super-canonical conformal weights. If the hypothesis $\Delta = \zeta^{-1}(z)$ is accepted, the radial conformal weight corresponds to a zero of Riemann Zeta: $\Delta = s_k$ at quantum criticality.

At quantum level a necessary prerequisite for the transition to occur is that radial conformal weights, which are conserved quantum numbers for the partonic time evolution, satisfy the constraint $\Delta = s_k$. The partonic 2-surfaces appearing in the vertices defining S-matrix elements for the phase transitions in question need not be of the required kind. It is enough that $\Delta = s_k$ condition allows their evolution to any sector of H in question. An analogous argument applies also to the phase transitions changing CP_2 Planck constant.

Quantum criticality for high temperature super-conductivity [J1, J2, J3] could provide an application for this vision. The super conducting stripe like regions are assumed to carry Cooper pairs with a large value of M^4 Planck constant corresponding to $n = 2^{11}$. The boundary region of the stripe is assumed to carry Cooper pairs in critical phase so that super-canonical conformal weights of electrons should satisfy $\Delta = s_k$ in this region. If the members of Cooper pair have conjugate conformal weights, the reality of super-canonical conformal weight is guaranteed. The model predicts that the critical region has thickness $L(151)$ whereas scaled electron with $n = 2^{11}$ effectively correspond to $L(127 + 22) = L(149)$, the thickness of the lipid layer of cell membrane.

These observations in turn lead to the hypothesis that cell interior corresponds to a phase with large M^4 Planck constant $\hbar(M^4) = 2^{11}\hbar_0$ and cell membrane to a quantum critical region where the above mentioned condition $\Delta = s_k$ is satisfied. Thus it would seem that the possibility of ordinary electron pairs to transform to large $\hbar$ Cooper pairs is essential in living matter and that the transition takes place as the electron pairs traverse cell membrane. The quantum criticality of cell membrane might prevail only in a narrow temperature range around T=37 C. Note that critical temperature range can also depend on the group G having C_n, $n = 2^{11}$ cyclic group as maximal cyclic group (C_n and D_n are the options).

7.3.6 What about zeros of Riemann poly-zeta?

As already found Riemann poly-zetas $\zeta_n(\Delta_1, ..., \Delta_n)$ could allow to generalize the notion of binding energy to that of binding conformal weight. In this case zeros form a continuum so that the set of points $(\Delta_1, ..., \Delta_n) = \zeta_n^{-1}(z = \xi^1/\xi^2)$ forms a n-1 complex dimensional surface in C^n. Completely symmetrized polyzetas are expressible using products of Riemann Zetas for arguments which are sums of arguments for polyzeta. If Δ_i are linear combinations of zeros of Zeta, polyzeta involves Riemann Zeta only for arguments which are sums of zeros of ζ. Symmetrized polyzeta is non-vanishing when Δ_i are non-trivial zeros of Zeta but vanishes for trivial zeros at $\Delta_i = -2n_i$. Also the zeros of symmetrized polyzeta would have interpretation in terms of quantum criticality.

An interesting question is whether ζ_n has a discrete subset of zeros for which p^{Δ_i} is algebraic number for all primes p and Δ_i. This could be the case. For instance, suitable linear combinations of zeros of ζ define zeros of polyzeta. For instance, $(a, b) = (s_1, s_1 - s_2)$ for any pair of zeros of zeta is zero of $P_2(a, b) = \zeta(a)\zeta(b) - \zeta(a+b)$ whereas $(a_1, a_2, a_3) = (s_1, s_2 - s_1, \overline{s}_2 - s_1)$ defines a zero of

$$P_3(a_1, a_2, a_3) = 2\zeta(a_1 + a_2 + a_3) + \zeta(a_1)P_2(a_2, a_3) + \zeta(a_2)P_2(a_3, a_1)$$
$$+\zeta(a_3)P_2(a_1, a_2) - 2\zeta(a_1)\zeta(a_2)\zeta(a_3)$$

for any pair (s_1, s_2) of zeros of ζ.

The conditions state that all P_m:s, $m < n$ in the decomposition of P_n vanish separately. Besides this one a_k, say $a_1 = s_1$ must correspond to a zero of ζ. Same is true for the sum σa_k and sub-sums involving a_1. The number of conditions increases rapidly as n increases. In the case of P_4 the three triplets (a_1, a_i, a_j) must be of same form as $n = 3$ case and this allows only the trivial solution with say $a_4 = 0$. Thus it would seem that only $n = 2$ and $n = 3$ allow non-trivial solutions for which bound state conformal weights are expressible in terms of differences of zeros of Riemann ζ. What is nice that the linear combinations of these conformal multi-weights give total conformal weights which are linear combinations of zeros of zeta.

The special role of 2- and 3-parton states brings unavoidably in mind mesons and baryons and the fact that hadrons containing larger number of valence quarks have not yet been identified experimentally.

If conformal confinement holds true then physical particles have vanishing conformal weights. This would require that ordinary baryons and mesons have real conformal weights and cannot therefore correspond to this kind of states. One must however take the notion of conformal confinement very critically. The point is that the one-dimensional logarithmic plane waves $x^{1/2+iy}$ have unitary inner product with respect to the scaling invariant inner product defined by the integration measure dx/x. For this inner product, the real part of the conformal weight should be 1/2 as it indeed is for the solutions of the conditions. If this interpretation is correct, then hadrons would represent states with non-vanishing conformal weight.

If one accepts complex conformal weights one must have some physical interpretation for them. The identification of conjugation of zeros of zeta as charge conjugation does not look promising since it would not leave neutral pion invariant. Of course, critical configurations with real conformal weight are possible at least formally and would correspond to trivial zeros $s_2 = -2n$ of ζ but s_1 arbitrary zero. These configuration would not however define logarithmic plane waves.

Laser physics might come in rescue here. So called phase conjugate photons are known to behave differently from photons. I have already proposed that all particles possess phase conjugates in TGD Universe. Phase conjugation is identified as reversal of time arrow mapping positive energy particles to negative energy particles. At space-time level this would mean an assignment of time orientation to space-time sheet. This is consistent with the fact that energy momentum complex consists of vector currents rather than forming a tensor. The implication is that in S-matrix positive energy particles travelling towards geometric future are not equivalent with negative energy particle travelling towards geometric past. This is essential for the notions like remote metabolism and time mirror mechanism.

The precise definition of phase conjugation at quantum level has remained obscure. The identification of phase conjugation as conjugation for the zeros of Zeta looks however very natural.

8 Can one formulate Quantum TGD as a quantum field theory of some kind?

The super-canonical generalization of a 2-dimensional conformal field theory seems to be indispensable for the construction of S-matrix at the fundamental level and defines the vertices as n-point functions of a conformal field theory in turn used to construct S-matrix as tree diagrams. It is however not at all obvious whether QFT like formulation in a more general sense really makes sense or that it is even needed.

Certainly it is clear that standard quantum field theory starting from an action principle and defining perturbation theory is out of question since it contradicts the basic assumption that S-matrix elements reduce to tree diagrams. This leaves two options.

1. The action of the possibly existing field theory limit must correspond to an effective action for one-particle irreducible (1PI) Green's functions from which Green's functions and S-matrix elements are obtained as tree diagrams. 1PI Greens function must correspond to the vertices identifiable as n-point functions of a conformal field theory for which super-canonical algebra defines the conformal fields.

2. If quantum TGD allows a formulation as a QFT, the formulation must be such that it effectively reduces to a free field theory. This is required by the vanishing of loops implying localization in the configuration space also necessary for the p-adicization of the theory.

8.1 Could one formulate quantum TGD as a quantum field theory at the absolute minimum space-time surface?

The original naive belief when I started to develop TGD for 25 years ago was that something like functional integral using EYM action coupled to induced spinor fields would define the quantum theory. The belief turned out to be wrong and the conclusion was that this kind of approach might make sense only as a quantum field theory limit of TGD. The philosophy just described and the view that space-time physics represents only classical correlates for the underlying configuration space physics however makes even the idea about the existence of quantum field theory limit questionable.

On the other hand, one could defend the existence of some kind of QFT type formulation by quantum classical correspondence and by the fact that the phenomenology based on the notion of classical induced gauge fields has been extremely fruitful. One could even hope that the theory would allow formulation as some kind of field theory at $X^4(X^3)$ or even better, at X^3 so that a minimum amount of information about the absolute minimum would be needed in accordance with the gravitational hologram principle.

8.1.1 Does the modified Dirac action for the induced spinor fields define the QFT description of quantum TGD?

The constraints satisfied by the QFT formulation of quantum TGD are so strong that the formulation is essentially unique.

1. The QFT in question should be determined by the absolute minimum $X^4(X^3)$ of Kähler action corresponding to a given causal determinant (7-dimensional light like surface $X^3_l \times CP_2$) rather than in M^4. The averaging over all Poincare and color translates of this space-time surface would give rise to an S-matrix respecting the basic symmetries. Note that the theory would be 3-D quantum field theory at X^3. The only information needed about $X^4(X^3)$ would be the time derivatives of the imbedding space coordinates at X^3. Also the value of Kähler action seems to be needed but even the exponent of Kähler function might disappear from the Greens functions in normalization just as the exponent e^G of the generating functional G of connected Green's functions disappears in the quantum field theory.

   The effective 3-dimensionality has an interesting connection to unresolved difficulties encountered in the attempt to formulate bound state problems in quantum field theory context. Non-relativistic, essentially 3-dimensional, Schrödinger equation works and yield correct predictions whereas Bethe-Salpeter equation in Minkowski space fails. The TGD based explanation of this failure is that bound state formation means that 3-surfaces of particles involved form a join along boundaries condensate. This means that non-relativistic 3-dimensional formulation is necessary in order to catch the essential aspects of the physics involved. In quantum field theory context point-likeness of the particles allows only the modelling of those aspects of particle interactions which do not involve bound states.

2. To calculate correlation functions one should expand the super-canonical generators as functional Taylor series around the maximum of the Kähler function at the 3-surface X^3. If super-canonical generators can be regarded as functionals of the second quantized induced spinor field ψ, also the functional series with respect to ψ is needed. This would make it possible to evaluate the correlation functions perturbatively in terms of the tree diagrams defined by the effective action using bosonic and fermionic propagators defined by it. One would calculate M^4 Fourier transforms of the correlation functions and integration over Poincare translates would give Poincare invariant correlation functions.

3. The complete localization at configuration space level means the vanishing of the bosonic loops and effective freezing of configuration space degrees of freedom. This is achieved if the action is such that the bosonic part vanishes when the induced spinor fields vanish. I have represented in [B4] arguments that the action for the induced spinor fields treated as Grassman variables is all that is needed to define quantum physics. Kähler action would be the effective action associated with the Dirac action. The approach would also predict the possible values of Kähler coupling strength as part of data characterizing the effective action. This approach also conforms with the fact that elementary bosons are predicted to be bound states of fermion-anti-fermion pairs.

   This approach would also bring induced electro-weak gauge potentials into play so that quantum-classical correspondence would be realized. The absence of quark color as spin like quantum number of induced spinor fields would not be a problem. Also the topologization of the family replication phenomenon in terms of the genus of 2-surface would result without representation as an additional spin like degeneracy of fermion fields. The super-canonical and super Kac-Moody conformal algebras would be however an essential element of the picture.

4. The action would be the modified Dirac action for the induced spinor fields at the maximum of the Kähler function. Modified Dirac action is defined by replacing the induced gamma matrices $\Gamma_\alpha = \partial_\alpha h^k \Gamma_k$ by the modified gamma matrices

$$\hat{\Gamma}^{\alpha} = \frac{\partial(L\sqrt{|g|})}{\partial(\partial_{\alpha}h^k)}\Gamma^k \quad . \tag{101}$$

Here L denotes the action density of Kähler action and Γ^k denotes gamma matrices of the imbedding space H. Modified Dirac action is supersymmetric and shares the vacuum degeneracy of Kähler action [B4].

The fermionic propagator would be defined by the inverse of the modified Dirac operator. Bosonic kinetic term would be Grassmann algebra valued and vanish for $\psi = 0$ and would contribute nothing to the perturbation series. Since the fermionic action is free action, a divergence free quantum field theory would be in question irrespective of whether the fermionic action is interpreted as an action or an effective action. One could also see the action as a fixed point of the map sending action to effective action in a complete accordance with the idea that loop corrections vanish.

5. The nice feature of this approach is that the information needed about the absolute minimum would be minimal since one can restrict the consideration to 3-surface X^3 which can be selected arbitrarily. In fact, the outcome is a 3-dimensional free field theory in the fermionic degrees of freedom and the integral over Poincare and color translates guarantees isometry symmetries. Grassmannian functional integral would give exponent of Kähler function as the analog of the generating functional G for connected Green functions. The functional derivatives of the configuration space spinor field with respect to the induced spinor field are very simple by the conservation of fermion number. This approach could be seen as an alternative approach to calculate n-point functions by treating fermionic fields as Grassmann fields whereas in the super-algebra approach fermionic fields would be second quantized.

8.1.2 Vacuum extremals and the fractality of super-canonical propagator

For vacuum extremals the propagator defined by the modified Dirac operator becomes singular. The interpretation as a space-time correlate for the singular behavior of the scalar propagator defined as a partition function in super-canonical algebra is suggestive.

Two different models for the scalar propagator were considered.

1. In the first case all physical super-canonical conformal weights were assumed to contribute and the resulting propagator has besides the ordinary pole infinite number of delta function type resonances. The real inverse of the propagator is poorly defined at the continuum limit. The mass squared values of the resonances are transcendental numbers and do not contribute at all to propagation assuming that the virtual momentum square values are rational or algebraic numbers. Therefore all p-adicizations of the propagator are trivial.

2. For the second option only the "holomorphic" sub-algebra of conformal weights contributes to the partition function. In this case a very complex spectrum of resonances, which are now poles, emerges. Also now singularities correspond to transcendental values of mass squared so that singular S-matrix elements are avoided when rational cutoff is used. The inverse of the propagator is well-defined but at the limit when all non-trivial zeros of Riemann Zeta are included both the propagator and itse inverse extremely singular mathematically. Obviously the kinetic part of the local field theory action for M^4 field theory would be very awkward at this limit and would not provide anything new.

The only manner to avoid filling the entire phase space with resonances, is to assume a physical (rather than only fictive) hierarchy of p-adic length scale and phase resolution cutoffs defined by the hierarchy of sub-algebras of the super-canonical algebra defined by the first n non-trivial zeros of Riemann Zeta.

As noticed, the ordinary perturbation theory fails for the vacuum extremals of Kähler action since the modified Dirac operator vanishes in this case identically. One might expect that the propagator develops a lot of singularities in the vicinity of vacuum extremals. Caustics would be the analog in the classical Maxwell's theory so that these singularities are expected to be very important physically. This singular behavior could correspond to the predicted presence of a dense set of singularities (delta function or pole singularities depending on the option) for the super-canonical propagator at the limit when all zeros of Zeta are included. If this were the case the limit as the cutoff N for the number of non-trivial zeros of Riemann Zeta included goes to infinity, would correspond to the approach to a vacuum extremal. Physically this would mean that zero modes would approach to a limit at which configuration space metric becomes degenerate. In particular, the dimension of CP_2 projection would approach $D \leq 2$ since Lagrange sub-manifold is in question. The sub-algebras of super-canonical algebra would classify the phases represented by space-time sheets and also conformal equivalence classes of the configuration space metric. Super-canonical algebra has infinite number of infinite-dimensional sub-algebras and each could correspond to a vacuum extremals. Note that the partition function hypothesis would effectively mean the explicit calculation of the n-point functions from symmetry considerations so that QFT limit would only provide the interpretation in terms of space-time physics rather than being a practical tool.

The zeros of Riemann Zeta are known to be associated with chaotic systems and vacuum extremals indeed correspond to chaotic systems by their non-deterministic behavior. The interpretation of the vacuum extremals as space-time correlates for engineered world (recall the non-determinism) conforms with this picture. The freedom to engineer would apply even to the mass spectrum of particles realized as a freedom to choose the sub-algebra of the super-canonical algebra.

8.2 Could field theory limit defined in M^4 or H be useful?

The general vision behind quantum TGD suggests that the field theory limit should not contain any dependence on the details of space-time surfaces. Thus it should be defined in either M^4 or $M^4 \times CP_2$. The theory would be defined by an effective action defining the vertices and propagators and tree diagrams would give the n-point functions and S-matrix elements. No problems with loop divergences would be encountered.

At least formally one could start from vertices and propagators defined by n-point functions of the conformal field theory as function of four-momenta and other quantum numbers of the incoming particles and interpret them as local vertices involving differential operators in Minkowski space. Assuming that the proper 2-point functions $\Gamma^{2)}$ identifiable as the inverse of the connected two point function $G_c^{2)}$ exist, one could interpret them as analogs of the operators defining the differential operators defining the kinetic part of the effective action. The construction of the scalar propagator as a partition function for the super-canonical algebra however suggests that the $\Gamma^{2)}$ fails to exist at the continuum limit without cutoff for the number of the zeros. As already described, an entire hierarchy of these functions involving physical length scale and phase resolution cutoffs is involved and has interpretation in terms of approach to vacuum extremal.

When the propagators defined as super-canonical are defined for finite sub-algebras of a holomorphic super-canonical algebra, the propagator and its inverse are well defined in momentum space. It is not obvious whether the construction of a local field theory in M^4 by using the inverse brings in anything interesting.

The cautious conclusion would be that effective field theory limit makes sense in M^4 only by taking the propagators and vertices defined as n-point functions of the conformal field theory and using momentum space Feynman rules to build tree diagrams. The QFT limit could be seen as low energy approximation obtained by taking into account the vertices involving light particles.

9 Appendix A: Some examples of bi-algebras and quantum groups

The appendix summarizes briefly the simplest bi- and Hopf algebras and some basic constructions related to quantum groups.

9.1 Simplest bi-algebras

Let $k(x_1,..,x_n)$ denote the free algebra of polynomials in variables x_i with coefficients in field k. x_i can be regarded as points of a set. The algebra $Hom(k(x_1,...,x_n),A)$ of algebra homomorphisms $k(x_1,...,x_n) \rightarrow A$ can be identified as A^n since by the homomorphism property the images $f(x_i)$ of the generators $x_1,...x_n$ determined the homomorphism completely. Any commutative algebra A can be identified as the $Hom(k[x],A)$ with a particular homomorphism corresponding to a line in A determined uniquely by an element of A.

The matrix algebra $M(2)$ can be defined as the polynomial algebra $k(a,b,c,d)$. Matrix multiplication can be represented universally as an algebra morphism Δ from from $M_2 = k(a,b,c,d)$ to $M_2^{\otimes 2} = k(a',a'',b',b'',c',c'',d',d'')$ to $k(a,b,c,d)$ in matrix form as

$$\Delta \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = \left(\begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) \left(\begin{array}{cc} a'' & b'' \\ c'' & d'' \end{array} \right) \ .$$

This morphism induces algebra multiplication in the matrix algebra $M_2(A)$ for any commutative algebra A.

$M(2)$, $GL(2)$ and $SL(2)$ provide standard examples about bi-algebras. $SL(2)$ can be defined as a commutative algebra by dividing free polynomial algebra $k(a,b,c,d)$ spanned by the generators a,b,c,d by the ideal $det-1 = ad-bc-1 = 0$ expressing that the determinant of the matrix is one. In the matrix representation μ and η are defined in obvious manner and μ gives powers of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ .$$

Δ, counit ϵ, and antipode S can be written in case of $SL(2)$ as

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ,$$

$$\begin{pmatrix} \epsilon(a) & \epsilon(b) \\ \epsilon(c) & \epsilon(d) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \ .$$

Note that matrix representation is only an economical manner to summarize the action of Δ on the generators a, b, c, d of the algebra. For instance, one has $\Delta(a) = a \to a \otimes a + b \otimes c$. The resulting algebra is both commutative and co-commutative.

$SL(2)_q$ can be defined as a Hopf algebra by dividing the free algebra generated by elements a, b, c, d by the relations

$$\begin{array}{rr} ba = qab \ , & db = qbd \ , \\ ca = qac \ , & dc = qcd \ , \\ bc = cb \ , & ad - da = (q^{-1} - 1)bc \ , \end{array}$$

and the relation

$$det_q = ad - q^{-1}bc = 1$$

stating that the quantum determinant of $SL(2)_q$ matrix is one.

$\mu, \eta, \Delta, \epsilon$ are defined as in the case of $SL(2)$. Antipode S is defined by

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \ .$$

The relations above guarantee that it defines quantum inverse of A. For q an n^{th} root of unity, $S^{2n} = id$ holds true which signals that these parameter values are somehow exceptional. This result is completely general.

Given an algebra, the R point of $SL_q(2)$ is defined as a four-tuple (A, B, C, D) in R^4 satisfying the relations defining the point of $SL_q(2)$. One can say that R-points provide representations of the universal quantum algebra $SL_q(2)$.

9.2 Quantum group $U_q(sl(2))$

Quantum group $U_q(sl(2))$ or rather, quantum enveloping algebra of $sl(2)$, can be constructed by applying Drinfeld's quantum double construction (to avoid confusion note that the quantum Hopf algebra associated with $SL(2)$ is the quantum analog of a commutative algebra generated by powers of a 2×2 matrix of unit determinant).

The commutation relations of $sl(2)$ read as

$$[X_+, X_-] = H \ \ , \ \ [H, X_\pm] = \pm 2X_\pm \ . \tag{102}$$

$U_q(sl(2))$ allows co-algebra structure given by

$$\begin{array}{ll} \Delta(J) = J \otimes 1 + 1 \otimes J \ , \ \ S(J) = -J \ , & \epsilon(J) = 0 \ , \ \ J = X_\pm, H \ , \\ S(1) = 1 \ , & \epsilon(1) = 1 \ . \end{array} \tag{103}$$

The enveloping algebras of Borel algebras $U(B_\pm)$ generated by $\{1, X_+, H\}$ $\{1, X_-, hH\}$ define the Hopf algebra H and its dual $H^\star$ in Drinfeld's construction. h could be called Planck's constant vanishes at the classical limit. Note that $H^\star$ reduces to $\{1, X_-\}$ at this limit. Quantum deformation parameter q is given by $exp(2h)$. The duality map $\star : H \rightarrow H^\star$ reads as

$$\begin{array}{l} a \rightarrow a^\star \ \ , \ \ ab = (ab)^\star = b^\star a^\star \ \ , \\ 1 \rightarrow 1 \ \ , \ \ \ H \rightarrow H^\star = hH \ \ , \ \ \ X_+ \rightarrow (X_+)^\star = hX_- \ . \end{array} \tag{104}$$

The commutation relations of $U_q(sl(2)$ read as

$$[X_+, X_-] = \tfrac{q^H - q^{-H}}{q - q^{-1}} \ \ , \ \ [H, X_\pm] = \pm 2X_\pm \ . \tag{105}$$

Co-product Δ, antipode S, and co-unit ϵ differ from those $U(sl(2))$ only in the case of $X_\pm$:

$$\begin{array}{l} \Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm \ \ , \\ S(X_\pm) = -q^{\pm 1} X_\pm \ . \end{array} \tag{106}$$

When q is not a root of unity, the universal R-matrix is given by

$$R = q^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(1-n)}{2}} q^{\frac{nH}{2}} X_+^n \otimes q^{-\frac{nH}{2}} X_-^n \ . \tag{107}$$

When q is m:th root of unity the q-factorial $[n]_q!$ vanishes for $n \geq m$ and the expansion does not make sense.

For q not a root of unity the representation theory of quantum groups is essentially the same as of ordinary groups. When q is m^{th} root of unity, the situation changes. For $l = m = 2n$ n^{th} powers of generators span together with the Casimir operator a sub-algebra commuting with the whole algebra providing additional numbers characterizing the representations. For $l = m = 2n+1$ same happens for m^{th} powers of Lie-algebra generators. The generic representations are not fully reducible anymore. In the case of $U_q(sl(2))$ irreducibility occurs for spins $n < l$ only. Under certain conditions on q it is possible to decouple the higher representations from the theory. Physically the reduction of the number of representations to a finite number means a symmetry analogous to a gauge symmetry. The phenomenon resembles the occurrence of null vectors in the case of Virasoro and Kac Moody representations and there indeed is a deep connection between quantum groups and Kac-Moody algebras [19].

One can wonder what is the precise relationship between $U_q(sl(2)$ and $SL_q(2)$ which both are quantum groups using loose terminology. The relationship is duality. This means the existence of a morphism $x \rightarrow \Psi(x)$ $M_q(2) \rightarrow U_q^\star$ defined by a bilinear form $\langle u, x\rangle = \Psi(x)(u)$ on $U_q \times M_q(2)$, which is bi-algebra morphism. This means that the conditions

$$\langle uv, x\rangle = \langle u\otimes v, \Delta(x)\rangle \ , \quad \langle u, xy\rangle = \langle \Delta(u), x\otimes y\rangle \ ,$$

$$\langle 1, x\rangle = \epsilon(x) \ , \qquad\qquad \langle u, 1\rangle = \epsilon(u)$$

are satisfied. It is enough to find $\Psi(x)$ for the generators $x = A, B, C, D$ of $M_q(2)$ and show that the duality conditions are satisfied. The representation

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \ \rho(K = q^H) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \ ,$$

extended to a representation

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

of arbitrary element u of $U_q(sl(2)$ defines for elements in $U_q^\star$. It is easy to guess that $A(u), B(u), C(u), D(u)$, which can be regarded as elements of $U_q^\star$, can be regarded also as R points that is images of the generators a, b, c, d of $SL_q(2)$ under an algebra morphism $SL_q(2) \rightarrow U_q^\star$.

9.3 General semisimple quantum group

The Drinfeld's construction of quantum groups applies to arbitrary semi-simple Lie algebra and is discussed in detail in [19]. The construction relies on the use of Cartan matrix.

Quite generally, Cartan matrix $A = \{a_{ij}\}$ is $n \times n$ matrix satisfying the following conditions:

i) A is indecomposable, that is does not reduce to a direct sum of matrices.

ii) $a_{ij} \leq 0$ holds true for $i < j$.

iii) $a_{ij} = 0$ is equivalent with $a_{ij} = 0$.

A can be normalized so that the diagonal components satisfy $a_{ii} = 2$.

The generators e_i, f_i, k_i satisfying the commutations relations

$$\begin{array}{ll} k_ik_j = k_jk_i \ , & k_ie_j = q_i^{a_{ij}}e_jk_i \ , \\ k_if_j = q_i^{-a_{ij}}e_jk_i \ , & e_if_j - f_je_i = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \ , \end{array} \tag{108}$$

and so called Serre relations

$$\begin{array}{l} \sum_{l=0}^{1-a_{ij}}(-1)^l \left[\begin{array}{c} 1-a_{ij} \\ l \end{array} \right]_{q_i} e_i^{1-a_{ij}-l}e_je_i^l = 0, \ \ i \neq j \ , \\ \sum_{l=0}^{1-a_{ij}}(-1)^l \left[\begin{array}{c} 1-a_{ij} \\ l \end{array} \right]_{q_i} f_i^{1-a_{ij}-l}f_jf_i^l = 0 \ , \ \ i \neq j \ . \end{array} \tag{109}$$

Here $q_i = q^{D_i}$ where one has $D_ia_{ij} = a_{ij}D_i$. $D_i = 1$ is the simplest choice in this case.

Comultiplication is given by

$$\Delta(k_i) = k_i \otimes k_i \ , \tag{110}$$

$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i \ , \tag{111}$$

$$\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes 1 \ . \tag{112}$$

(113)

The action of antipode S is defined as

$$S(e_i) = -e_i k_i^{-1} \ , \quad S(f_i) = -k_i f_i \ , \quad S(k_i) = -k_i^{-1} \ . \tag{114}$$

9.4 Quantum affine algebras

The construction of Drinfeld and Jimbo generalizes also to the case of untwisted affine Lie algebras, which are in one-one correspondence with semisimple Lie algebras. The representations of quantum deformed affine algebras define corresponding deformations of Kac-Moody algebras. In the following only the basic formulas are summarized and the reader not familiar with the formalism can consult a more detailed treatment can be found in [19].

1. Affine algebras

The Cartan matrix A is said to be of affine type if the conditions $det(A) = 0$ and $a_{ij}a_{ji} \geq 4$ (no summation) hold true. There always exists a diagonal matrix D such that $B = DA$ is symmetric and defines symmetric bilinear degenerate metric on the affine Lie algebra.

The Dynkin diagrams of affine algebra of rank l have $l+1$ vertices (so that Cartan matrix has one null eigenvector). The diagrams of semisimple Lie-algebras are sub-diagrams of affine algebras. From the $(l+1) \times (l+1)$ Cartan matrix of an untwisted affine algebra $\hat{A}$ one can recover the $l \times l$ Cartan matrix of A by dropping away 0:th row and column.

For instance, the algebra A_1^1, which is affine counterpart of $SL(2)$, has Cartan matrix a_{ij}

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

with a vanishing determinant.

Quite generally, in untwisted case quantum algebra $U_q(\hat{G}_l)$ as $3(l+1)$ generators e_i, f_i, k_i $(i = 0, 1, .., l)$ satisfying the relations of Eq. 109 for Cartan matrix of $\mathcal{G}^{(1)}$. Affine quantum group is obtained by adding to $U_q(\hat{\mathcal{G}}_l)$ a derivation d satisfying the relations

$$[d, e_i] = \delta_{i0} e_i \ , \quad [d, f_i] = \delta_{i0} f_i, \quad [d, k_i] = 0 \ . \tag{115}$$

with comultiplication $\Delta(d) = d \otimes 1 + 1 \otimes d$.

2. Kac Moody algebras

The undeformed extension $\hat{\mathcal{G}}_l$ associated with the affine Cartan matrix $\mathcal{G}_l^{(1)}$ is the Kac Moody algebra associated with the group G obtained as the central extension of the corresponding loop algebra. The loop algebra is defined as

$$L(\mathcal{G}) = \mathcal{G} \otimes C\left[t, t^{-1}\right] \ , \tag{116}$$

where $C\left[t, t^{-1}\right]$ is the algebra of Laurent polynomials with complex coefficients. The Lie bracket is

$$[x \times P, y \otimes Q] = [x, y] \otimes PQ \ . \tag{117}$$

The non-degenerate bilinear symmetric form $(,)$ in $\mathcal{G}_l$ induces corresponding form in $L(\mathcal{G}_l)$ as $(x \otimes P, y \otimes Q) = (x, y)PQ$.

A two-cocycle on $L(\mathcal{G}_l)$ is defined as

$$\Psi(a, b) = Res(\frac{da}{dt}, b) \ , \tag{118}$$

where the residue of a Laurent is defined as $Res(\sum_n a_n t^n) = a_{-1}$. The two-cocycle satisfies the conditions

$$\begin{aligned} \Psi(a, b) &= -\Psi(b, a) \ , \\ \Psi([a, b], c) + \Psi([b, c], a) + \Psi([c, a], b) &= 0 \ . \end{aligned} \tag{119}$$

The two-cocycle defines the central extension of loop algebra $L(\mathcal{G}_l)$ to Kac Moody algebra $L(\mathcal{G}_l) \otimes Cc$, where c is a new central element commuting with the loop algebra. The new bracket is defined as $[,] + \Psi(,)c$. The algebra $\tilde{L}(\mathcal{G}_l)$ is defined by adding the derivation d which acts as td/dt measuring the conformal weight.

The standard basis for Kac Moody algebra and corresponding commutation relations are given by

$$\begin{aligned} J_n^x &= x \otimes t^n \ , \\ [J_n^x, J_m^y] &= J_{n+m}^{[x,y]} + n\delta_{m+n,0}c \ . \end{aligned} \tag{120}$$

The finite dimensional irreducible representations of G defined representations of Kac Moody algebra with a vanishing central extension $c = 0$. The highest weight representations are characterized by highest weight vector $|v\rangle$ such that

$$\begin{aligned} J_n^x|v\rangle &= 0, \ \ n > 0 \ , \\ c|v\rangle &= k|v\rangle \ . \end{aligned} \tag{121}$$

3. Quantum affine algebras

Drinfeld has constructed the quantum affine extension $U_q(\mathcal{G}_l)$ using quantum double construction. The construction of generators uses almost the same basic formulas as the construction of semi-simple algebras. The construction involves the automorphism $D_t : U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right] \rightarrow U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right]$ given by

$$\begin{array}{ll} D_t(e_i) = t^{\delta_{i0}} e_i \ , & D_t(f_i) = t^{\delta_{i0}} f_i \ , \\ D_t(k_i) = k_i & D_t(d) = d \ , \end{array} \tag{122}$$

and the co-product

$$\Delta_t(a) = (D_t \otimes 1)\Delta(a) \ , \quad \Delta_t^{op}(a) = (D_t \otimes 1)\Delta^{op}(a) \ , \tag{123}$$

where the $\Delta(a)$ is the co-product defined by the same general formula as applying in the case of semi-simple Lie algebras. The universal R-matrix is given by

$$\mathcal{R}(t) = (D_t \otimes 1)\mathcal{R} \ , \tag{124}$$

and satisfies the equations

$$\begin{aligned} &\mathcal{R}(t)\Delta_t(a) = \Delta_t^{op}(a)\mathcal{R} \ , \\ &(\Delta_z \otimes id)\mathcal{R}(u) = \mathcal{R}_{13}(zu)\mathcal{R}_{23}(u) \ , \\ &(id \otimes \Delta_u)\mathcal{R}(zu) = \mathcal{R}_{13}(z)\mathcal{R}_{12}(zu) \ , \\ &\mathcal{R}_{12}(t)\mathcal{R}_{13}(tw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(tw)\mathcal{R}_{12}(t) \ . \end{aligned} \tag{125}$$

The infinite-dimensional representations of affine algebra give representations of Kac-Moody algebra when one restricts the consideration to generations e_i, f_i, k_i, $i > 0$.

10 Appendix B: Riemann Zeta and propagators

One might hope that TGD possesses a well-defined and divergence free quantum field theory limit. The preceding considerations suggest a manner to get rid of the divergences of quantum field theories. Super-canonical conformal weights, or more generally, zeros of Riemann Zeta, should leave a trace in quantum field theory and this trace must be essential for the cancellation of infinities. In the sequel an attempt to understand how the zeros of Zeta might be visible at single particle level is made. This attempt should be taken as a mere mathematical experimentation having rather ad hoc character and it does not represent the main bulk of TGD.

10.1 General model for a scalar propagator

10.1.1 Is the p-adic length scale evolution of propagators universal and determined by the zeros of zeta

The quantum field theory model for polyzetas [22] gives some hints about what generalized Feynman diagrams could be.

1. In the 1-dimensional quantum field theory model for polyzetas discussed in [22] the propagator factors correspond to all possible integer powers G^k of a propagator G defined as inverse of the derivative operator $D = d/dt$. The resulting diagrams give results which are proportional to polyzetas. The integer k is completely analogous to a conformal weight. For a single loop approximation of the scalar field theory only the power D^2 would appear as a propagator and this would give only polyzetas with even weight.

2. The work of Kreimer and Broadhurst [23] leads to the conclusion that one can assign to each vacuum Feynman diagram a closure of a unique braid with chords connecting the strands of the braid: this is nothing but the TGD counterpart of vacuum diagram defined in terms of a braid with chords representing propagators. Probably this correspondence holds for non-vacuum diagrams also. These braids give rise to

combinations of polyzetas such that the number n of strands corresponds to the depth $m = n + 1$ of the polyzeta and the number of crossings equals to the weight $k = \sum_i k_i$ of the polyzeta.

3. The polyzeta associated with the Feynman diagram vanishes if the allowed conformal weights k_i satisfying $\sum k_i = k$ are not positive integers but correspond to the zeros of polyzeta in question. Hence the replacement of the integers k_i identified as conformal weights with the zeros of the polyzeta in question gives hopes that one could understand the p-adic length scale evolution of the propagator and the cancellation of loop corrections. The physical interpretation for the modified conformal weights would as "bound state conformal weights", and the necessity to use them would be something which field theory alone cannot explain. The form of the propagator to be discussed is completely universal and applies also to stringy diagrams when one replaces propagator with the Super Kac-Moody conformal scaling generator L_0.

10.1.2 Various models for scalar propagator

The first thing that comes in mind is to construct a model for the scalar field propagator based based on zeros of ζ. The are several options.

Option I: This model is inspired by the quantum field theoretical model for polyzetas [22]. Scalar field propagator is regarded as a kind of partition function in an ensemble whose states are labelled by spectrum of super-canonical weights expressible in terms of zeros of Zeta. This assumption is admittedly somewhat adhoc. A sum over both the non-trivial and trivial zeros is needed and trivial zeros give the usual free propagator contribution. The details of this option are left to Appendix B.

Option II: The inspiration comes now from the formula $\int_0^\infty exp(i(p^2 - m^2 - i\epsilon)t)dt$ for the scalar propagator. The integral is replaced with a discrete sum over allowed values of $t > 0$ which on basis of number theoretical considerations are argued to correspond to linear combinations for the imaginary parts of non-trivial zeros of Zeta with coefficients which are positive integers to guarantee $t > 0$.

Option III: This model is inspired by the observation that the defining integral can be transformed to an integral over positive imaginary axis in t-plane and one can wonder whether it could be replaced with a sum over the trivial zeros of Zeta. This would give a rather realistic looking propagator.

Option IV: One can criticize the notion of the momentum space scalar propagator as being a too phenomenological concept to allow the application of notions of super-canonical invariance and quantum criticality.

1. In TGD the usual momentum space propagators can have only a phenomenological role since 4-momentum does not flow in the lines of the generalized Feynman diagrams. The counterpart of the propagator originates from the unitary evolution for the induced spinor fields at 3-D light-like causal determinants representing orbits of 2-D partons along which generalized eigen modes of the modified Dirac operator propagate. Momentum squared as the argument of the propagator is replaced by the square of the eigen value of the modified Dirac operator D.

2. There are reasons to suspect [C7] that the unitary evolution associated with the lines of generalized Feynman diagrams corresponds to the unitary evolution operator Δ^{it} determined by the positive Hermitian "Hamiltonian" operator Δ inherent for von Neumann algebras. Positivity suggests that Δ corresponds to the square D^2 of the modified Dirac operator. Conformal invariance would in turn suggest that the eigenvalues of Δ and possibly also D^2 might relate in a simple manner to the positive integer valued spectrum of Virasoro generator L_0. In this case the propagator would result through a summation over the number theoretically allowed discrete values of t as for option II. The propagator reduces to a function which can be regarded as a dual of Riemann Zeta.

10.1.3 General conditions on scalar propagator

Consider as an example a scalar field theory with the momentum space inverse propagator $D = p^2 - m^2$.

1. Introduce a mass scale λ playing the role of an ultraviolet cutoff and replace the inverse propagator $D_0 \equiv \lambda^{-2}D$ with a conformal weight $k = -2$ with the sum of propagators with conformal weights z which correspond to the zeros of ζ. In the quantum field theory context the parameter λ would be an ad hoc parameter but in TGD framework its values naturally corresponds to the inverses of p-adic length scales so that one would have entire hierarchy of propagators with $\lambda \propto 1/\sqrt{p}$, p prime and discretized renormalization group evolution.

2. In the standard renormalization theory one poses the conditions

$$G_R^{-1}(p^2 = m^2) = 0 \ , \quad \frac{dG_R^{-1}}{dp^2}(p^2 = m^2) = 1 \ . \tag{126}$$

These conditions are posed also now. In the standard renormalization theory the propagator has a cut along real axis above $p^2 = m^2$ and the cut is due to the logarithmic terms $log[(p^2 - m^2)/\lambda^2]$ whose imaginary part is discontinuous. One might expect this behavior also now.

For options II and III the definition of scalar propagator is unique. Concerning the precise definition of the propagator for option I, one can consider several options.

1. The Cartan decomposition $g = h + t$ of the super-canonical algebra discussed in [B2] determines the conformal weights in question and they correspond to the conformal weights of t. This option has a good theoretical justification since the generators of t indeed corresponding to non-gauge degrees of freedom. In this case also conformal weights of form $z = n - 1/2 - \sum_i y_i$, where y_i are imaginary parts of the non-trivial zeros of Riemann zeta must be included. The resulting propagator has an interpretation as a propagator with ultraviolet cutoff. The restriction $n = 0, 1$ emerges from the requirement that the propagator is non-singular on mass shell. This option is implied also by the requirement that Virasoro generators L_n, $n \geq 1$, act as gauge symmetries in the super-canonical algebra.

2. One can also consider the restriction to the conformal sub-algebra of the super-canonical algebra so that only the linear combinations $\sum n(y)y$, $n(y) > 0$, are involved.

3. The minimal option is that only the conformal weights corresponding to the zeros of Riemann Zeta contribute to the scalar propagator.

The character of the non-trivial zeros of Riemann Zeta is of crucial physical significance.

1. The assumption that the imaginary parts of the zeros are linearly independent is attractive from the calculational point of view. It however turns that the resulting propagator has either delta function like poles or infinitely narrow resonance poles ro real mass mass squared values and this is not a physically realistic result although it could be a good

approximation. Furthermore the propagator is totally ill defined at the limit when all zeros of Zeta contribute since the singularities are dense on real axis.

2. The distribution of the zeros of Zeta favors strongly the option for which zeros are not linearly independent and that the phases p^{iy} for each prime p correspond to sums of subset of angles of Pythagorean prime phases Φ_P defined by the squares of Gaussian primes and a fixed set of rational fractions $m2\pi$. This implies that the generating super-canonical conformal weights are not linearly independent. With this assumption the singularities of the propagator are not anymore at real values of mass squared and the propagator is well-defined also at the limit when all zeros of Zeta contribute to it.

In the following calculations the assumption that y_i are linearly independent is made although this assumption is probably wrong. The motivation is that it allows to understand the behavior of the propagator approximately and see why the linear independence implied by the ansatz inspired by the properties of the spectral function of Riemann zeros is physically highly desirable.

The hypothesis that p^{iy} can be expressed in terms of Pythagorean prime phase angle and rational multiple of 2π for any prime, if true, is bound to be a very powerful number theoretic symmetry. In particular, it gives infinite number of representations of the scalar field propagator, one for each prime. In the sequel this aspect is not discussed at all.

10.1.4 Propagators for various options

1. Scalar field propagator for option II

In this case the expression for scalar field propagator is easy to derive if linear independence of y_i is assumed. The expression for the propagator reads as

$$G = \sum_{n_i>0} u^{i\sum_i n_i y_i} \ , \tag{127}$$

$$u = exp(\frac{p^2-m^2}{\lambda^2}) \ . \tag{128}$$

Here the sum over the combinations of zeros with positive integer valued coefficients is assumed. One could assume that only the condition $\sum n_i y_i > 0$

holds true. The restriction is consistent with the hypothesis about the p-adic existence of basic building blocks at zeros of Riemann Zeta if $exp((p^2 - m^2)/\lambda^2)$ is restricted to be rational.

Performing the sums, one obtains

$$
\begin{aligned}
G(p) &= \prod_i \frac{1}{1-u^{iy_i}} \ , \\
u &= exp(\frac{p^2-m^2+i\epsilon}{\lambda^2}) \ .
\end{aligned} \tag{129}
$$

This expression can be regarded as a dual of Riemann Zeta in the sense that the product over partition functions associated with primes is replaced with the product over partition functions labelled by non-trivial zeros. The terms of product approach to unity with exponential rate for $\epsilon > 0$ so that the product converges.

The propagator has poles at

$$
p^2 = m^2 + k\lambda^2\frac{2\pi}{y_i} \ , \ k \in Z \ . \tag{130}
$$

$k = 0$ corresponds to $p^2 = m^2$ so that scalar propagator pole is obtained. The mass formula is stringy mass formula with string tension proportional to $1/y_i$.

If u corresponds stringy propagator $L_0^{tot} = (p^2 - L_0)/\lambda^2$ instead of scalar propagator one obtains modification of the stringy mass formula

$$
m^2 = (n + k\frac{2\pi}{y_i})\lambda^2 \ , \ k \in Z \ . \tag{131}
$$

If all non-trivial zeros of ζ are allowed, the poles are dense in the entire m^2-axis so that also tachyons are obtained. One can consider several cures the problem.

1. Assuming that the basic building blocks of Riemann Zeta exist p-adically for its zeros, the poles with $k > 0$ do not correspond to rational values of u so that the poles other than $p^2 = m^2$ ($p^2 = n\lambda^2$ in the stringy case) are not at all visible if the theory is required to satisfy p-adicizability constraint.

2. A natural hierarchy of cutoffs is provided by the subalgebras of super-canonical algebra defined by y-cutoff. In this case poles are not dense anymore but tachyons remain.

3. If y_i are not linearly independent, it is not possible to sum freely over integers n_i and poles are smoothed out and are expected to develop imaginary parts. If the condition $n_i > 0$ is replaced with the weaker condition $\sum_i n_i y_i > 0$ more correlates between y_i emerge and further smoothing out of poles is expected to happen.

2. Scalar field propagator for option III

In this case the propagator is given by as sum over trivial zeros of ζ at $z = -2n$.

$$G(u) = iu^2/(1-u^2) = i\frac{exp(\frac{2(p^2-m^2)}{\lambda^2})}{1-exp(\frac{2(p^2-m^2)}{\lambda^2})} \quad . \qquad (132)$$

The pole at $p^2 = m^2$ is not shifted and for large values of $p^2 > 0$ the propagator approaches -1 and vanishes exponentially for space-like momenta.

3. Scalar propagator for option I

There are several alternatives concerning definition of the scalar propagator as a partition function. These are discussed in detail in Appendix B. What makes this option questionable is that the pole at m^2 is shifted to $m^2 + \lambda^2$. This would require that massless particles result from tachyons. The other problems and possible cures are same as those for option II. One can say that the number theoretic discretization of the integral formula for propagator is favored over the partition function based model.

4. Propagator for option IV

In this case the interpretation as momentum space propagator is given up. The general expression of propagator is same as for option II: only the argument $u = exp((p^2 - m^2)/\lambda^2)$ is replaced by a new one. A good guess is that u corresponds to the squares for the eigenvalues of the modified Dirac operator required to be rational by p-adicization. An alternative guess is that u corresponds to the non-negative integer valued spectrum of Virasoro generator L_0.

If y_i are linearly independent and $n_i > 0$ condition is assumed one obtains poles also now but they do not correspond rational values of u. The values of poles would be given by

$$u_{k,y_i} = exp(\frac{2\pi k}{y_i}) \ . \tag{133}$$

Rationality requirement, the linear independence of y_i, the weakened condition $\sum_i n_i y_i > 0$, and already the fact that the spectrum of D^2 is predetermined, excludes poles. It seems that option III favored also by the general physical picture is the only realistic one.

In the following various options for scalar field propagator defined as a partition function are studied in detail.

10.2 Scalar field propagator for option I

In the following various options for scalar field propagator defined as a partition function are studied in detail for option I.

10.2.1 Propagator assuming all conformal weights predicted by super-canonical algebra

The super-canonical algebra defining the algebra of isometries of configuration space has conformal weights expressible in terms of both trivial and non-trivial zeros of Riemann Zeta and this leads to the idea that scalar field propagator could be regarded as a partition function in super-canonical algebra. The construction turned out to be extremely useful by providing the physical input making it easier to grasp the structure of the super-canonical algebra. In the following the assumption that y_i are linearly independent is made in the hope that it is a good approximation although the facts about the spectral function favor the ansatz predicting linear dependence.

1. Scalar propagator as a partition function for Cartan decomposition of the super-canonical algebra

The idea is to construct propagator as a partition function for the super-canonical algebra and require that it is faithful to the Cartan decomposition of the algebra defining configuration space of 3-surfaces as a symmetric space. As shown in [B2], the super-canonical algebra decomposes to two parts $g = g_t + g_{nt}$ corresponding to trivial and non-trivial zeros of Zeta.

1. The Cartan decomposition $g = t + h$, $[t,t] \subset h$, $[h,h] \subset h$, $[t,h] \subset t$ means that only the generators of t correspond to physical degrees of freedom because of coset space-structure meaning that h corresponds to gauge degrees of freedom. Thus only the conformal weights of t

contribute to the partition function defining the propagator. The conformal weights associated with t_n are $h_t = 2n$ whereas the conformal weights associated with t_{nt} are

$$\begin{aligned} h_{nt,1} &= 2n - \tfrac{1}{2} - i\textstyle\sum_{y_i>0} n_i y_i \ , \ \sum_i n_i = 2N+1 \ , \ n>0 \ , \\ h_{nt,2} &= 2n + \tfrac{1}{2} - i\textstyle\sum_{y_i>0} n_i y_i \ , \ \sum_i n_i = 2N \ , \ n>0 \ . \end{aligned} \tag{134}$$

The sub-spaces defined by $n=0$ defined together with trivial zeros an orthogonal basis. It is however unclear whether $n>0$ states can satisfy the orthogonality conditions and whether they should be excluded as states generated by Virasoro generators L_{2n}, $n>0$.

2. The propagator is defined as a partition function

$$\begin{aligned} \frac{G_R}{\lambda^2} &= \sum_{z\in t} u^{f(z)} \ , \\ u &= \frac{p^2-m^2}{\lambda^2} \ , \end{aligned} \tag{135}$$

where $f(z)$ is a linear function of the conformal weight $z = h_t, ht_{tn}$.

3. Propagator can be expressed as a sum of three parts

$$\begin{aligned} \frac{G_R}{\lambda^2} &= G_{R,1} + G_{R,2} + G_{R,3} \\ &= \epsilon_1 \sum_{z=-2n} G_0^{f(z)} + \epsilon_2 \sum_{z=h_{nt,1}} G_0^{f(z)} + \epsilon_3 \sum_{z=h_{nt,2}} G_0^{f(z)} \ . \end{aligned} \tag{136}$$

The sign factors must be chosen in such a manner that the resulting expression for the propagator is physically acceptable. The sign factor should define a Lie algebra homomorphism: $\epsilon([A,B]) = \epsilon(A)\epsilon B$. For the choice $\epsilon_1 = \epsilon_3 = 1, \epsilon_2 = \pm 1$ this condition is satisfied. In the following $\epsilon = \epsilon_3 = 1$ is assumed and the value of ϵ_2 is left free: it turns out that $\epsilon_2 = -1$ is the only possible consistent choice if one allows all values of n for conformal weights hnt. If only $n = 0,1$ is allowed then $\epsilon_2 = 1$ is possible.

4. The requirement that $1/(p^2-m^2)$ behavior results as the lowest order contribution from $G_{R,1}$, implies $f(z)=-z/2$, and one has

$$\begin{aligned}
\frac{G_{R,1}}{\lambda^2} &= \sum_{n>0} G_0^n \ , \\
\frac{G_{R,2}}{\lambda^2} &= \epsilon_2 \sum_n G_0^{2n-1/2} \times \sum_{\sum n(y)y} G_0^{\left[-i\sum n(y)y\right]/2}{}_{|\sum_y n(y)\ odd} \ , \\
\frac{G_{R,3}}{\lambda^2} &= \sum_n G_0^{2n+1/2} \times \sum_{\sum n(y)y} G_0^{\left[-i\sum n(y)y\right]/2}{}_{|\sum_y n(y)\ even} \ . \qquad (137)
\end{aligned}$$

Here $\sum n(y)y$ appearing as summation index can be regarded a formal superposition over imaginary parts of arbitrary subset of non-trivial zeros of Zeta (note that the linear independence over zeros is an essential although probably only approximate assumption).

Consider the expressions for the various parts of the propagator.

1. The expression for $G_{R,1}$ reads as

$$G_{R,1} = \frac{1}{p^2-m^2-\lambda^2} \ . \qquad (138)$$

The pole is shifted unless $m^2(\lambda)$ depends on λ as $m^2=m_0^2-\lambda^2$.

2. The expressions for $G_{R,2}$ and $G_{R,3}$ are sums of terms proportional to sums

$$\sum_{\sum_i n(y_i)y_i} exp\left[-ilog(u)\sum_i n(y_i)y_i/2\right]$$

where $n(y)$ is odd or even. For $u>0$ the conditions guaranteeing the vanishing of these sums can be derived by assuming that the imaginary parts y of the zeros of Riemann Zeta are linearly independent do not satisfy any identify of form

$$\sum_i n_i y_i = 0 \ , \ \ n_i \in Z \ ,$$

so that they generate an Abelian group with an infinite number of generators defined by y. With this assumption the sum over each y gives a delta function $\delta(log(u)-4\pi k/y)$, and if the exponent contains more than one y_i, the result is zero. The sums give delta functions allowing p^2 to have only discrete values

$$p^2 = m_{y,k}^2 = m^2 + \lambda^2 \times e^{\frac{2\pi k}{y}} \ , \ k \in Z \ , \ y > 0 \ . \tag{139}$$

The expressions for $G_{R,2}$ and $G_{R,3}$ are

$$\begin{aligned} \frac{G_{R,2}}{\lambda^2} &= \epsilon_2 \sum_{k,y>0} \delta(u-u_{k,y})(-1)^k u_{k,y}^{-1/4} f_i(u_{k,y}) \ , \\ \frac{G_{R,3}}{\lambda^2} &= \sum_{k,y>0} \delta(u-u_{k,y}) u_{k,y}^{1/4} f_i(u_{k,y}) \ , \\ u_{k,y} &= \frac{m_{k,y}^2 - m^2}{\lambda^2} = e^{\frac{2\pi k}{y_i}} \ . \end{aligned} \tag{140}$$

The function $f_i(u_{k,y})$, $i = 1,2$ results from the summation over powers u^n, and $i = 1,2$ labels $n = 0,1$ and $n = 0,1,...$ options for the conformal weights h_{nt}. One has

$$\begin{aligned} f_1(u_{k,y}) &= 1 + u(k,y) \ , \\ f_2(u_{k,y}) &= \frac{1}{1-u(k,y)} \ . \end{aligned} \tag{141}$$

Except for the dependence coming from the delta functions expressing the fact that resonances masses scale as λ^2, these parts of propagator are independent of λ.

3. The overall expression for the propagator in the region $p^2 - m^2 \geq 0$ reads as

$$\frac{G_R}{\lambda^2} = \frac{1}{p^2 - m^2 - \lambda^2} + \sum_{k,y>0} \delta(p^2 - m_{k,y}^2) \left[u_{k,y}^{1/4} + \epsilon_2 (-1)^k u_k^{-1/4} \right] f_i(u_{k,y}) \ . \tag{142}$$

4. The guess that the propagator in the space-like region $p^2 - m^2 < 0$ is obtained by an analytical continuation turns out to be correct. The explicit summation gives zero always irrespective of the value of whether u is positive or negative unless the resonance conditions are satisfied. This means that resonance contributions vanish for $p^2 - m^2 < 0$, that is for $p^2 < m_0^2 - \lambda^2$ for $m^2 = m_0^2 - \lambda^2$ option implied by renormalization group invariance.

2. Pole and resonance structure of the propagator

The pole of the propagator is shifted to $m^2 + \lambda^2$ and renormalization conditions are satisfied at $p^2 = m^2 + \lambda^2$. The renormalization group invariance condition applied on mass shell gives $\lambda dG_R/d\lambda = 0$. Only $G_{R,1}$ contributes to the condition for on mass shell and one has

$$m^2(\lambda) = m_0^2 - \lambda^2 \quad . \tag{143}$$

Obviously $m^2 = m_0^2$ becomes renormalization group invariant.

The propagator is thus a sum of free propagator with additional delta-function contributions at $p^2 = m^2_{y_i,k}$ for $k \neq 0$,. There appears no pole (that is propagating physical particle) at these masses because one has $[p^2 - m^2_{y_i,k}]\delta(p^2 - m^2_{y_i,k}) = 0$.

Consider now the choice of the parameter ϵ_2 and the choice between $n = 0, 1$ and $n < 0$ options corresponding to $f_1(u) = 1 + u$ and $f_2(u) = 1/(1-u)$.

1. For $n = 0, 1$ option corresponding to $f_1(u) = 1 + u$ no pole is generated for $k = 0$ and $\epsilon_2 = 1$ is possible. The choice $\epsilon_2 = -1$ however cancels the resonance at $p^2 = m_0^2$ corresponding to $k = 0$. This is highly desirable since at the limit when all zeros of Zeta are allowed the resonance strength of this resonance becomes infinite. The result conforms with the view that the generators L_n, $n \geq 1$, indeed generate zeros states.

2. For $n > 0$ option corresponding the situation is more delicate assuming that the condition $m^2(\lambda^2) = m_0^2 - \lambda^2$ is satisfied. The factor

$$\frac{\left[u_{k,y}^{1/4} + \epsilon_2(-1)^k u_{k,y}^{-1/4}\right]}{1 - u_{k,y}} = \frac{e^{\frac{\pi k}{2y}} + \epsilon_2(-1)^k e^{-\frac{\pi k}{2y}}}{1 - e^{\frac{2\pi k}{y}}}$$

would have a pole for $\epsilon_2 = 1$ because of vanishing of the denominator $1 - u_{k=0,y}$ and the residue of the propagator would be infinite at $p^2 =$

m^2 rather than being equal to 1. Thus the only possible choice is $\epsilon_2 = -1$ in this case. One has however still the problem with the infinite value of the resonance strength when all zeros of Zeta are included.

One can say that the virtual mass squared possesses a discrete resonance spectrum superposed to the free propagation. The interpretation as a kind of fractal noise might make sense. The propagator contains the resonance contributions also in the space-like region for $p^2 - m^2 = m_0^2 - \lambda^2 < 0$.

The mass spectrum for the resonances can be written as

$$m_{k,y}^2 = m_0^2 + \lambda^2 \left[e^{\frac{2\pi k}{y}} - 1 \right] \ , \tag{144}$$

and depends on λ. The spectrum contains both s-channel resonances corresponding to $k > 0$ and tachyonic exchange resonances corresponding to $k < 0$. The symmetry between the two spectra brings in mind string model duality. For tree diagrams the mass quantization of exchanged resonances would mean that discretization occurs in the momentum space for resonances. For instance, in the scattering of two particles in cm frame the space-like resonances appear at discrete values of the scattering angle θ as a kind of a fractal rainbow effect. The couplings to the resonances are reduced exponentially at the time-like limit $k \to \infty$ and increase exponentially at the space-like limit $k \to -\infty$ ($m^2 \to m_0^2 - \lambda^2$).

The inverse $D = 1/G$ of the propagator does not make sense at the momenta corresponding to the resonances. The reason is that the geometric series expansion of

$$G_R^{-1} = \frac{1}{G_{R,3}[1 + G_{R,3}^{-1}(G_{R,1} + G_{R,2})]}$$

involves ill-defined powers of delta functions $\delta(p^2 - m_{k,y_i}^2)$. The interpretation is that the propagator does not allow a non-singular field theory limit.

3. Why a cutoff in the number of non-trivial zeros of Riemanm Zeta is necessary?

Any subset $\{y > 0\}$ of non-trivial zeros of Riemann Zeta defines a sub-algebra of the super-canonical algebra. This suggests a hierarchy of sub-algebras defined by the set $Y_n = \{y_1, ..., y_n\}$ of n lowest zeros defining a hierarchy of propagators containing only sum over conformal weights of a sub-algebra of super-canonical algebra. The restriction to sub-algebra defined by Y_n could be interpreted as a cutoff in y-space. This cutoff would not be a mere calculational trick but would represent infinite hierarchy of "phases" with a finite y-cutoff meaning increasing density of points in the lattice defined by the set Y_n. One could even think that the full super-canonical algebra as well as propagators, vertices and S-matrices are *defined* by a nested hierarchy of sub-algebras defined by the subsets $Y_n \subset Y_{n+1}$.

There are several reasons making this kind of cutoff necessary.

1. Without the cutoff the number of resonances in a mass squared interval $[m^2, m^2 + \Delta m^2]$ is infinite. The condition for the resonances can be written as

$$\frac{m^2 - m_0^2}{\lambda^2} + 1 = exp\left[\frac{2\pi k}{y}\right] \quad .$$

   For sufficiently large values of y one can always find at least one $k(y)$ such that the resulting mass squared is contained in the interval $[m^2, m^2 + \Delta m^2]$. Without the y-cutoff rates for the scattering with final state momenta restricted in a finite volume elements of the phase space would be infinite. The situation changes for rational cutoff since the resonances turn out to correspond to the values of u which are not rational or do not even belong to a finite-dimensional extension of rationals. In this case the delta singularities disappear.

2. The p-adicization of the theory requires y-cutoff. Consider p-adicization in a given p-adic number field R_p. Suppose that the values of $u = (p^2 - m^2)/\lambda^2$ appearing in the propagator are rational. The p-adic existence of the phase factors u^{iy} is required by the definition of the partition function as a function making sense also p-adically. If rationals containing primes smaller than some cutoff prime p_c are allowed, this condition boils down to the existence of the phases q^{iy}, $q \leq p_c$ prime, as a number in a finite-dimensional extension of the p-adic number field R_p.

   Suppose that the phase factors q^{iy}, q prime belong to a finite-dimensional algebraic extension of any p-adic number field R_p for any prime q and any non-trivial zero $z = 1/2 + y$ of Zeta. This is achieved if the phases p^{iy_i} are expressible as products of roots of unity and Pythagorean phases:

$$
\begin{aligned}
p^{iy} &= e^{i\phi_P(p,y)} \times e^{i\phi(p,y)} \ , \\
e^{i\phi_P(p,y)} &= \frac{r^2 - s^2 + i2rs}{r^2 + s^2} \ , \ r = r(p,y) \ , \ s = s(p,y) \ , \\
e^{i\phi(p,y)} &= e^{i\frac{2\pi m}{n}} \ , \ m = m(p,y) \ , \ n = n(p,y) \ . \qquad (145)
\end{aligned}
$$

The Pythagorean phases associated with two different zeros of zeta must be different in order to have linear independence y_i over integers. Of course, linear independence is just a working hypothesis to be tested and it will be found that there is support for the view that linear independence leads to non-physical results and is not favored by the facts known from the distribution of zeros of Zeta. Pythagorean phases form a multiplicative group having the phases of squares of Gaussian primes as its generators. The squares of Gaussian primes are primes $p \ mod \ 4 = 1$.

One can decompose for any p any zero y_i into two parts $y_i = y_{ia} + y_{iP}$ such that one has

$$
log(p)y_{ia}(p) = \frac{m2\pi}{n} \ , \ log(p)y_{iP} = arctan\left[\frac{2rs}{r^2+s^2}\right] \ , \qquad (146)
$$

where m, n, r, s of course depend on y and p. The decomposition is unique if one requires $m < n$.

3. The exponents $exp(\pi k/2y)$ appearing in the formula for the propagator can be expressed as

$$
e^{\frac{\pi k}{2y}} = q^{\frac{k}{2[log(p)y_1(p) + y_P(p)]}} \ . \qquad (147)
$$

Due to the presence of y_P these numbers are not rational nor even algebraic unless additional conditions are posed. Scalar propagator would reduce to a free propagator for rational values of u propagator and the p-adicization of the propagator would always give just the free propagator.

4. The propagator could be formally interpreted as a partition function with the conformal weights of t being in the role of energy spectrum and $2log(u)$ in the role of the inverse temperature. In the real context physical high energy limit corresponds to a formal low temperature limit. The p-adic length scale hierarchy with decreasing p-adic length scales and increasing value of λ presumably corresponds to a series of phase transitions increasing λ and the temperature in a discontinuous manner occur unless one can achieve overcooling. p-Adically the increase of the upper bound for u means that the set of rational values of u increases in size. This means the increase of p_c and thus also of the p-adic prime p.

5. The propagator has several non-physical features. The reduction to single particle sector in the sense that all contributions involving sums of at least two different zeros y vanish, is definitely a pathological feature. The presence of delta function singularities for real mass squared values is a second un-physical feature. The limit when all zeros contribute to the propagator gives real propagator for which delta function singularities fill the real axis. The reduction to a free propagator in p-adic context for all rational cutoffs is a further questionable feature. All these features follow from the assumption that the zeros are linearly independent. The actual linear dependence implied by the ansatz explaining the correlations between zeros implies that this approximation leads to multiple counting of some super-canonical weights and there are good hopes that the singularities are due to this over counting.

10.2.2 Propagator as a partition function associated with the holomorphic sub-algebra of the super-canonical algebra

The interpretation of the conformal weights of super-canonical algebra as punctures of complex plane at which Super Kac-Moody conformal Virasoro algebra acts as conformal transformations encourages to consider also the subalgebra generated by allowing only the generators with conformal weights $z = -1/2 - iy$, $y > 0$ and $z = n$, $n > 0$. In this case the sums appearing in the definition of the propagator as a partition function would be only over sums $\sum_n (y)n(y)y$ involving only positive values of y and one would have $N > 0$ and $n(y) \geq 0$ in the sums of Eq. 137. Also now the working hypothesis is that y_i are linearly independent.

The outcome is the following general formula

$$\frac{G_{R,2}}{\lambda^2} = \epsilon_2 u^{-1/4} f_i(u) \times \sum_{N\geq 0} \sum_{\sum n(y)y} X\left[\sum n(y)y|2N+1\right] ,$$

$$\frac{G_{R,3}}{\lambda^2} = u^{1/4} f_i(u) \times \sum_{N\geq 1} \sum_{\sum n(y)y} X\left[\sum n(y)y|2N\right] ,$$

$$X\left[\sum n(y)y|M\right] = u^{\left[-i\sum n(y)y\right]/2} , \sum n(y) = M. \quad (148)$$

Here $\sum_y n(y)y$ refers to formal sum of non-trivial zeros of Zeta with $n(y) > 0$. The terms in the sum can be arranged according to how many different zeros contribute to the sum $\sum n(y)y$, denote this number by K. The sum can be decomposed to a sum over all possible different subsets $U_K = \{y_{i_1}, ..., y_{i_K}\}$ of non-trivial zeros of Zeta. Hence one can write

$$\frac{G_{R,2}}{\lambda^2} = \epsilon_2 u^{-1/4} f_i(u) \times \sum_{K\geq 1} \sum_{U_K} \sum_{N\geq K-1} \sum_{\sum n_k = 2N+1} e^{-i\sum_{k=1}^K n_k y_{i_k} log(u)} ,$$

$$\frac{G_{R,3}}{\lambda^2} = u^{1/4} f_i(u) \times \sum_{K\geq 1} \sum_{U_K} \sum_{N\geq K-1} \sum_{\sum n_k = 2N} e^{-i\sum_{k=1}^K n_k y_{i_k} log(u)} . \quad (149)$$

Here $n_k \geq 1$ holds true for every y_{i_k}. The evaluation of the sums in terms of geometric sums is in principle straightforward for arbitrary value of K but the expressions get complicated as K increases.

Apart from factors $u^{\pm 1/4} f_i(u)$, $G_{R,2}$ *resp.* $G_{R,3}$ could be interpreted as a partition function for a system consisting of harmonic oscillators with fundamental frequencies y_i with the restriction that the total number of quanta is odd *resp.* even for the allowed states.

This modification alters dramatically the structure of the propagator.

1. Terms involving sums $\sum_k n_k y_{i_k}$ in the exponent contribute to the propagator and the outcome is terms depending on arbitrary many different y:s instead of only single particle terms as in the previous case.

2. Single particle contributions are transformed from resonances to genuine poles. For instance, in $G_{R,2}$ the sum

$$\sum_{n(y)\in Z} e^{ilog(u)(2n_y+1)/2}$$

giving single particle contribution is replaced with the sum

$$\sum_{n(y)\geq 0} e^{ilog(u)(2n(y)+1)/2} = \frac{e^{ilog(u)y/2}}{1-e^{ilog(u)y}} \quad .$$

The delta functions at $u = exp(k2\pi/y)$ transform to poles. This predicts infinite number of particles.

$n = 0, 1$ option with $\epsilon_2 = -1$ seems to be the only sensible choice since in this case there is no singularity at $p^2 = m_0^2$. This option guarantees the vanishing of $G_{R,2} + G_{R,3}$ at $u = 1$ quite generally. For other options the pole contribution at $u = 1$ ($p^2 = m_0^2$) is present, and implies that the propagator normalization is changed and without y-cutoff the residue of the propagator at $p^2 = m_0^2$ is infinite.

Tachyons result unless the condition

$$m_0^2 > \lambda^2$$

holds true so that λ^2 would play the role of infrared- rather than ultraviolet cutoff. This raises the possibility that the restriction to holomorphic sub-algebra makes sense for infrared cutoff whereas full algebra would make sense for ultraviolet cutoff. Ultraviolet cutoff means finite time resolution $\tau \sim 1/\sqrt{\lambda}$ and the interpretation would be that the lifetimes of resonances are shorter than τ. Ultraviolet cutoff would also mean that multiple y states are not seen at all in the spectrum. By previous arguments delta function type singularities disappear completely form the p-adicized theory whereas poles are visible but for rational values of u they do not produce diverging S-matrix elements.

3. Two-particle contributions read as

$$\frac{1}{e^{iy_2x} - e^{iy_1x}} \times \left[\frac{e^{i4y_2x}}{1-e^{i2y_2x}} - \frac{e^{i4y_1x}}{1-e^{i2y_1x}} \right] \quad , \quad x = log(u) \quad .$$

   Besides new poles poles at $u = e^{k\pi/y_i}$ there are also poles at $u = e^{k2\pi/(y_1-y_2)}$.

4. The study of three-particle contributions suggests that the term containing K different y:s gives rise to poles at $u = e^{k2\pi/x}$, for $x = 2y_i$ and $x = y_{ij} = y_i - y_j$ and $xi \neq j \neq k = y_{ij} + y_{kj}$, etc.

5. In order to not fill the entire phase space densely with propagator poles one must assume a hierarchy of y-cutoffs. The most general hierarchy is obtained by restricting the consideration to a set of finitely generated subalgebras defined by the sets U_K, $K < K_{max}$. In this case the convergence is not guaranteed, in particular in p-adic context there could be problems since infinite number of terms are present. Number-theoretical considerations suggest a simpler filtered hierarchy of subalgebras defined by the sets $\{y_1, ...y_K\}$.

6. Also this propagator has non-physical features. The presence of singularities are real values of mass squared is physically unrealistic feature since resonance widths are infinitely narrow in this kind of situation. The propagator is ill-define at the limit when all zeros contribute to it. These features are solely due to the assumed linear independence of y_i.

10.2.3 Propagator assuming that only zeros of ζ contribute to the spectrum of conformal weights

Without any idea about the structure of the super-canonical algebra the most probable guess would be that the partition function defining the scalar propagator contains only the contribution of trivial and non-trivial zeros but not the contributions corresponding to the commutators of these generators. This option could be also defended by saying that the conformal weights $\pm 1/2 + \sum_i n_i y_i$ correspond in some sense to many particle states.

With this assumption the propagator would reduce to a sum of two terms analogous to $G_{R,1}$ and $G_{R,2}$ in the general formula with the summations restricted to $n(y) = 1$. The term corresponding to $G_{R,3}$ would not be present since $G_{R,3}$ can involve only even values of $\sum n(y)$. This would lead to a propagator with a resonance type on mass shell singularity. In the following the option allowing $n(y) = 2$ at the line $Re(h) = 1/2$ is also considered since it gives a non-singular propagator at $p^2 = m_0^2$.

From the general formula of Eq. 137 for the propagator one finds that the expression for the propagator in this case reads as

$$G_R = \frac{1}{p^2 - m^2 - \lambda^2} + \epsilon_2 u^{-1/4} \sum_{y>0} cos\left[ilog(u)\frac{y}{2}\right] + \delta \times u^{1/4} \sum_{y>0} cos\left[ilog(u)y\right] \quad . \tag{150}$$

$\delta = 0$ characterizes the strictly "single particle" case $n(y) = 1$.

The requirement of on mass shell renormalization group invariance implies $m^2 = m_0^2 - \lambda^2$. ($\delta = 1, \epsilon_2 = -1$) option allows a propagator having only a pole singularity at $p^2 = m_0^2$ and satisfying renormalization conditions. For $\delta = 0$ the additional resonance singularity is unavoidable but renormalization conditions are still satisfied. The propagator reads as

$$\begin{aligned} G_R &= \frac{1}{p^2 - m_0^2} - (1 + \frac{p^2 - m_0^2}{\lambda^2})^{-1/4} \sum_{y>0} cos\left[ilog(1 + \frac{p^2 - m_0^2}{\lambda^2})\frac{y}{2}\right] \\ &+ \delta \times (1 + \frac{p^2 - m_0^2}{\lambda^2})^{1/4} \sum_{y>0} cos\left[ilog(1 + \frac{p^2 - m_0^2}{\lambda^2})y\right] . \end{aligned} \tag{151}$$

The powers of the logarithm imply a cut for $p^2 < m_0^2 - \lambda^2$ whereas in the region $p^2 > m_0^2 - \lambda^2$ logarithm and $1/4^{th}$ root have power series expansion. It is not clear whether the sum over zeros converges or not in the general case.

The sum over infinite number of phase factors u^{iy} is problematic p-adically.
i) The basic hypothesis is that the phase factors p^{iy} belong to finite-dimensional extensions of p-adic numbers for all primes p and zeros of Zeta. This would mean that the u^{iy} exists for all rational values of u but the sum $\sum_y u^{iy}$ fails to exist as an infinite sum of numbers possessing unit p-adic norm. y-cutoff would be necessary.
ii) If y would exist in an extension of R_p, cosines exist provided the conditions $|log\left[1 + (p^2 - m^2)/\lambda^2\right])y|_p < 1$ hold true. In the p-adic context the logarithm exists for $|(p^2 - m^2)/\lambda^2|_p < 1$ and has the same p-adic norm as $(p^2 - m^2)/\lambda^2$. In 2-adic case also the condition $|(p^2 - m^2)y/2\lambda^2|_2 < 1$ must be satisfied.

It is interesting to look what happens if one assumes that the hypothesis explaining the correlations between the zeros holds true. In this case the sum over zeros decomposes into a separate sum over Pythagorean phases y_P and over rational phases assumed to be same for each y_P. One can write the term

$$A \equiv \sum_{y>0} cos\left[ilog(1 + \frac{p^2 - m_0^2}{\lambda^2})\frac{y}{x}\right] ,$$

where one has either $x = 1$ or $x = 2$ as a real part of a product two sums:

$$q = Re(X) \ ,$$
$$X = \sum_{y_P>0} exp\left[ilog(1+\frac{p^2-m_0^2}{\lambda^2})\frac{\Phi_P}{xlog(p)}\right] \times \sum_{q\in Q_0} exp\left[ilog(1+\frac{p^2-m_0^2}{\lambda^2})\frac{q2\pi}{xlog(p)}\right] \ .$$

The first sum is over phase angles associated with a subset of Pythagorean prime phases (squares of Gaussian primes) and second over rational phase angles. Note that the non-uniqueness due to the decomposition to Pythagorean and algebraic phase angle does not affect the value of the X.

The conclusion is that the proposed form of the propagator is not favored by neither physical nor number theoretic conditions.

10.2.4 Do primes and their inverses correspond to the zeros of the propagators $G_{R,...}$?

The previous considerations rely on the probably wrong working hypothesis that the imaginary parts y_i of the nontrivial zeros $z_i = 1/2 + y_i$, $y_i > 0$, of Riemann Zeta are linearly independent.

The scalar propagator is defined as a partition function in the super-canonical algebra and one can decompose it into a sum of three parts with $G_{R,2}$ and $G_{R,3}$ being directly related to the non-trivial zeros of ζ. Also Riemann Zeta has an interpretation as a partition function. This raises interesting questions. Could Riemann Zeta and the building blocks $G_{R,2}$ and/or $G_{R,3}$ or some closely related partition functions be somehow duals of each other? Could Riemann Zeta be for ordinary primes what $G_{R,2}$ and/or $G_{R,3}$ are to y_i? What could be the counterpart of Riemann Zeta for the primes $z_i = 1/2 + y_i$?

If the correspondences makes sense, G_R or $G_{R,2}$ and/or $G_{R,3}$ should vanish for $u = p$ and $u = 1/p$ in an analogy with the fact that $1/2 \pm y_i$ and its conjugate are zeros of Zeta. This would predict the mass formula $M^2(p) = m_0^2 - \lambda^2 + p^{\pm 1}\lambda^2$. It is interesting to look whether this hypothesis might make sense in the approximation that the zeros are linearly independent.

1. In the case of the scalar propagator defined by all zeros of ζ the function $G_{R,2}+G_{R,3}$ has delta function peaks at transcendental values of u and vanishes trivially for $u = p^{\pm 1}$. $G_{R,1}$ has a pole for $u = 1$. This propagator would have as its analog the rational Zeta function $\zeta_q = \sum_q q^{-z}$, which is ill defined unless one poses cutoff by allowing only primes $p < p_c$. This cutoff would be analogous to the y-cutoff. One can however define ζ_q by noticing that it can be formally written as

$$\hat{\zeta}_q = \frac{1}{N}\sum_{m>0,n>0}(m/n)^{-z} = \frac{1}{N}\zeta(z)\zeta(-z) \ ,$$

where N is the number of all integers n. The expression results by writing $q = r/s = nr/ns$ and summing over all values of all positive integers n and dividing by the number N of all integers. By an infinite re-normalization one obtains a well-defined function $\zeta_q \equiv N \times \hat{\zeta}_q = \zeta(z)\zeta(-z)$. Interestingly, the inverse of this function has poles at the lines $Re(z) = \pm 1/2$ containing the super-canonical conformal weights.

2. The fact that Riemann Zeta is a partition function for the states labelled by integers suggests that the dual partition function should be a partition function for the states corresponding to "integers" defined by Riemann Zeta and thus correspond to the "holomorphic" partition function. It is clear from the expressions of $G_{R,2}$ and $G_{R,3}$ that if they vanish for u they vanish also for $1/u$. Also now the transcendental poles at $u = p^{k2\pi/y}$ are present as are also poles at $u = p^{k\pi/y}$: these poles should become complex poles when the linear dependence is taken into account. The points $u = p$ could be zeros but it is difficult to prove this. Note however that $G_{R,2}$ and $G_{R,3}$ do not directly correspond to ζ since only the sums of odd and even number of y_i:s are involved.

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