Recent Status of Lepto-Hadron Hypothesis

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Abstract

TGD suggests strongly the existence of lepto-hadron physics. Lepto-hadrons are bound states of color excited leptons and the anomalous production of e^+e^- pairs in heavy ion collisions finds a nice explanation as resulting from the decays of lepto-hadrons with basic condensate level $k = 127$ and having typical mass scale of one MeV. The recent indications on the existence of a new fermion with quantum numbers of muon neutrino and the anomaly observed in the decay of ortopositronium give further support for the lepto-hadron hypothesis. There is also evidence for anomalous production of low energy photons and e^+e^- pairs in hadronic collisions.

The identification of lepto-hadrons as a particular instance in the predicted hierarchy of dark matters interacting directly only via graviton exchange allows to circumvent the lethal counter arguments against the lepto-hadron hypothesis (Z^0 decay width and production of colored lepton jets in e^+e^- annihilation) even without assumption about the loss of asymptotic freedom.

PCAC hypothesis and its sigma model realization lead to a model containing only the coupling of the lepto-pion to the axial vector current as a free parameter. The prediction for e^+e^- production cross section is of correct order of magnitude only provided one assumes that lepto-pions decay to lepto-nucleon pair $e^+_{ex}e^-_{ex}$ first and that lepto-nucleons, having quantum numbers of electron and having mass only slightly larger than electron mass, decay to lepton and photon. The peculiar production characteristics are correctly predicted. There is some evidence that the resonances decay to a final state containing $n > 2$ particle and the experimental demonstration that lepto-nucleon pairs are indeed in question, would be a breakthrough for TGD.

During 18 years after the first published version of the model also evidence for colored μ has emerged. Towards the end of 2008 CDF anomaly gave a strong support for the colored excitation of τ. The lifetime of the light long lived state identified as a charged τ-pion comes out correctly and the identification of the reported 3 new particles as p-adically scaled up variants of neutral τ-pion predicts their masses correctly. The observed muon jets can be understood in terms of the special reaction kinematics for the decays of neutral τ-pion to 3 τ-pions with mass scale smaller by a factor 1/2 and therefore almost at rest. A spectrum of new particles is predicted. The discussion of CDF anomaly led to a modification and generalization of the original model for lepto-pion production and the predicted production cross section is consistent with the experimental estimate.

1 Introduction

TGD suggest strongly ('predicts' is perhaps too strong expression) the existence of color excited leptons. The mass calculations based on p-adic thermodynamics and p-adic conformal invariance lead to a rather detailed picture about color excited leptons.

1. The simplest color excited neutrinos and charged leptons belong to the color octets ν_8 and L_{10} and $L_{\overline{10}}$ decuplet representations respectively and lepto-hadrons are formed as the color singlet bound states of these and possible other representations. Electro-weak symmetry suggests strongly that the minimal representation content is octet and decuplets for both neutrinos and charged leptons.

2. The basic mass scale for lepto-hadron physics is completely fixed by p-adic length scale hypothesis. The first guess is that color excited leptons have the levels $k = 127, 113, 107, ...$ ($p \simeq 2^k$, k prime or power of prime) associated with charged leptons as primary condensation levels. p-Adic length scale hypothesis allows however also the level $k = 11^2 = 121$ in case of electronic lepto-hadrons. Thus both $k = 127$ and $k = 121$ must be considered as a candidate for the level associated with the observed lepto-hadrons. If also lepto-hadrons correspond non-perturbatively to exotic Super Virasoro representations, lepto-pion mass relates to pion

mass by the scaling factor $L(107)/L(k) = k^{(107-k)/2}$. For $k = 121$ one has $m_{\pi_L} \simeq 1.057$ MeV which compares favorably with the mass $m_{\pi_L} \simeq 1.062$ MeV of the lowest observed state: thus $k = 121$ is the best candidate contrary to the earlier beliefs. The mass spectrum of lepto-hadrons is expected to have same general characteristics as hadronic mass spectrum and a satisfactory description should be based on string tension concept. Regge slope is predicted to be of order $\alpha' \simeq 1.02/MeV^2$ for $k = 121$. The masses of ground state lepto-hadrons are calculable once primary condensation levels for colored leptons and the CKM matrix describing the mixing of color excited lepton families is known.

The strongest counter arguments against color excited leptons are the following ones.

1. The decay widths of Z^0 and W boson allow only $N = 3$ light particles with neutrino quantum numbers. The introduction of new light elementary particles seems to make the decay widths of Z^0 and W untolerably large.

2. Lepto-hadrons should have been seen in e^+e^- scattering at energies above few MeV. In particular, lepto-hadronic counterparts of hadron jets should have been observed.

A possible resolution of these problems is provided by the loss of asymptotic freedom in lepto-hadron physics. Lepto-hadron physics would effectively exist in a rather limited energy range about one MeV.

The development of the ideas about dark matter hierarchy [F6, F8, F9, J6] led however to a much more elegant solution of the problem.

1. TGD predicts an infinite hierarchy of various kinds of dark matters which in particular means a hierarchy of color and electro-weak physics with weak mass scales labelled by appropriate p-adic primes different from M_{89}: the simplest option is that also ordinary photons and gluons are labelled by M_{89}.

2. There are number theoretical selection rules telling which particles can interact with each other. The assignment of a collection of primes to elementary particle as characterizer of p-adic primes characterizing the particles coupling directly to it, is inspired by the notion of infinite primes [E3], and discussed in [F6]. Only particles characterized by integers having common prime factors can interact by the exchange of elementary bosons: the p-adic length scale of boson corresponds to a common primes.

3. Also the physics characterized by different values of $\hbar$ are dark with respect to each other as far quantum coherent gauge interactions are considered. Laser beams might well correspond to photons characterized by p-adic prime different from M_{89} and de-coherence for the beam would mean decay to ordinary photons. De-coherence interaction involves scaling down of the Compton length characterizing the size of the space-time of particle implying that particles do not anymore overlap so that macroscopic quantum coherence is lost.

4. Those dark physics which are dark relative to each other can interact only via graviton exchange. If lepto-hadrons correspond to a physics for which weak bosons correspond to a p-adic prime different from M_{89}, intermediate gauge bosons cannot have direct decays to colored excitations of leptons irrespective of whether the QCD in question is asymptotically free or not. Neither are there direct interactions between the QED:s and QCD:s in question if M_{89} characterizes also ordinary photons and gluons. These ideas are discussed and applied in detail in [F6, F8, F9].

Skeptic reader might stop the reading after these counter arguments unless there were definite experimental evidence supporting the lepto-hadron hypothesis.

1. The production of anomalous e^+e^- pairs in heavy ion collisions (energies just above the Coulomb barrier) suggests the existence of pseudoscalar particles decaying to e^+e^- pairs. A natural identification is as lepto-pions that is bound states of color octet excitations of e^+ and e^-.

2. The second puzzle, Karmen anomaly, is quite recent [18]. It has been found that in charge pion decay the distribution for the number of neutrinos accompanying muon in decay $\pi \rightarrow \mu+\nu_\mu$ as a function of time seems to have a small shoulder at $t_0 \sim ms$. A possible explanation is the decay of charged pion to muon plus some new weakly interacting particle with mass of order 30 MeV [19]: the production and decay of this particle would proceed via mixing with muon neutrino. TGD suggests the identification of this state as color singlet leptobaryon of, say type $L_B = f_{abc}L_8^aL_8^b\bar{L}_8^c$, having electro-weak quantum numbers of neutrino.

3. The third puzzle is the anomalously high decay rate of orto-positronium. [20]. e^+e^- annihilation to virtual photon followed by the decay to real photon plus virtual lepto-pion followed by the decay of the virtual lepto-pion to real photon pair, $\pi_L\gamma\gamma$ coupling being determined by axial anomaly, provides a possible explanation of the puzzle.

4. There exists also evidence for anomalously large production of low energy e^+e^- pairs [21, 22, 23, 24] in hadronic collisions, which might be basically due to the production of lepto-hadrons via the decay of virtual photons to colored leptons.

In this chapter a revised form of lepto-hadron hypothesis is described.

1. Sigma model realization of PCAC hypothesis allows to determine the decay widths of lepto-pion and lepto-sigma to photon pairs and e^+e^- pairs. Ortopositronium anomaly determines the value of $f(\pi_L)$ and therefore the value of lepto-pion-lepto-nucleon coupling and the decay rate of lepto-pion to two photons. Various decay widths are in accordance with the experimental data and corrections to electro-weak decay rates of neutron and muon are small.

2. One can consider several alternative interpretations for the resonances.

   Option 1: For the minimal color representation content, three lepto-pions are predicted corresponding to $8, 10, \overline{10}$ representations of the color group. If the lightest lepto-nucleons e_{ex} have masses only slightly larger than electron mass, the anomalous e^+e^- could be actually $e_{ex}^+ + e_{ex}^-$ pairs produced in the decays of lepto-pions. One could identify 1.062, 1.63 and 1.77 MeV states as the three lepto-pions corresponding to $8, 10, \overline{10}$ representations and also understand why the latter two resonances have nearly degenerate masses. Since d and s quarks have same primary condensation level and same weak quantum numbers as coloured e and μ, one might argue that also colored e and μ correspond to $k = 121$. From the mass ratio of the colored e and μ, as predicted by TGD, the mass of the muonic lepto-pion should be about 1.8 MeV in the absence of topological mixing. This suggests that 1.83 MeV state corresponds to the lightest $g = 1$ lepto-pion.

   Option 2: If one believes sigma model (in ordinary hadron physics the existence of sigma meson is not established and its width is certainly very large if it exists), then lepto-pions are accompanied by sigma scalars. If lepto-sigmas decay dominantly to e^+e^- pairs (this might be forced by kinematics) then one could adopt the previous sceneario and could identify 1.062 state as lepto-pion and 1.63, 1.77 and 1.83 MeV states as lepto-sigmas rather than lepto-pions. The fact that muonic lepto-pion should have mass about 1.8 MeV in the absence of topological mixing, suggests that the masses of lepto-sigma and lepto-pion should be rather close to each other.

   Option 3: One could also interpret the resonances as string model 'satellite states' having interpretation as radial excitations of the ground state lepto-pion and lepto-sigma. This identification is not however so plausible as the genuinely TGD based identification and will not be discussed in the sequel.

3. PCAC hypothesis and sigma model leads to a general model for lepto-hadron production in the electromagnetic fields of the colliding nuclei and production rates for lepto-pion and other lepto-hadrons are closely related to the Fourier transform of the instanton density $\bar{E}\cdot\bar{B}$ of the electromagnetic field created by nuclei. The first source of anomalous e^+e^- pairs is the production of $\sigma_L\pi_L$ pairs from vacuum followed by $\sigma_L \rightarrow e^+e^-$ decay. If $e^+_{ex}e^-_{ex}$ pairs rather than genuine e^+e^- pairs are in question, the production is production of lepto-pions from vacuum followed by lepto-pion decay to lepto-nucleon pair.

   Option 1: For the production of lepto-nucleon pairs the cross section is only slightly below the experimental upper bound for the production of the anomalous e^+e^- pairs and the decay rate of lepto-pion to lepto-nucleon pair is of correct order of magnitude.

   Option 2: The rough order of magnitude estimate for the production cross section of anomalous e^+e^- pairs via $\sigma_l\pi_l$ pair creation followed by $\sigma_L \rightarrow e^+e^-$ decay, is by a factor of order $1/\sum N_c^2$ (N_c is the total number of states for a given colour representation and sum over the representations contributing to the ortopositronium anomaly appears) smaller than the reported cross section in case of 1.8 MeV resonance. The discrepancy could be due to the neglect of the large radiative corrections (the coupling $g(\pi_L\pi_L\sigma_L) = g(\sigma_L\sigma_L\sigma_L)$ is very large) and also due to the uncertainties in the value of the measured cross section.

   Given the unclear status of sigma in hadron physics, one has a temptation to conclude that anomalous e^+e^- pairs actually correspond to lepto-nucleon pairs.

4. The vision about dark matter suggests that direct couplings between leptons and lepto-hadrons are absent in which case no new effects in the direct interactions of ordinary leptons are predicted. If colored leptons couple directly to ordinary leptons, several new physics effects such as resonances in photon-photon scattering at cm energy equal to lepto-pion masses and the production of $e_{ex}\bar{e}_{ex}$ (e_{ex} is leptobaryon with quantum numbers of electron) and $e_{ex}\bar{e}$ pairs in heavy ion collisions, are possible. Lepto-pion exchange would give dominating contribution to $\nu - e$ and $\bar{\nu} - e$ scattering at low energies. Lepto-hadron jets should be observed in e^+e^- annihilation at energies above few MeV:s unless the loss of asymptotic freedom restricts lepto-hadronic physics to a very narrow energy range and perhaps to entirely non-perturbative regime of lepto-hadronic QCD.

This chapter is a revised version of the earlier chapter [16] and still a work in progress. I apologize for the reader for possible inconvenience. The motivation for the re-writing came from the evidence for the production of τ-pions in high energy proton-antiproton collisions [59, 60]. Since the kinematics of these collisions differs dramatically from that for heavy ion collisions, a critical re-examination of the earlier model - which had admittedly somewhat ad hoc character- became necessary. As a consequence the earlier model simplified dramatically. As far as basic calculations are considered, the modification makes itself visible only at the level of coefficients. Even more remarkably, it turned out possible to calculate exactly the lepto-pion production amplitude under a very natural approximation, which can be also generalized so that the calculation of production amplitude can be made analytically in high accuracy and only the integration over lepto-pion momentum must be carried out numerically. As a consequence, a rough analytic estimate for the production cross section follows and turns out to be of correct order of magnitude. It must be however stressed that the cross section is highly sensitive to the value of the cutoff parameter (at least in this naive estimate) and only a precise calculation can settle the situation.

2 Lepto-hadron hypothesis

2.1 Anomalous e^+e^- pairs in heavy ion collisions

Heavy ion-collision experiments carried out at the Gesellschaft fur Schwerionenforschung in Darmstadt, West Germany [25, 26, 36, 37] have yielded a rather puzzling set of results. The expectation was that in heavy ion collisions in which the combined charge of the two colliding ions exceeds 173, a composite nucleus with $Z > Z_{cr}$ would form and the probability for spontaneous positron emission would become appreciable.

Indeed, narrow peaks of widths of roughly 50-70 keV and energies about 350± 50 keV were observed in the positron spectra but it turned out that the position of the peaks seems to be a constant function of Z rather that vary as Z^{20} as expected and that peaks are generated also for Z smaller than the critical Z. The collision energies at which peaks occur lie in the neighbourhood of 5.7-6 MeV/nucleon. Also it was found that positrons are accompanied by e^-- emission. Data are consistent with the assumption that some structure at rest in cm is formed and decays subsequently to e^+e^- pair.

Various theoretical explanations for these peaks have been suggested [27, 28]. For example, lines might be created by pair conversion in the presence of heavy nuclei. In nuclear physics explanations the lines are due to some nuclear transition that occurs in the compound nucleus formed in the collision or in the fragmets formed. The Z-independence of the peaks seems however to exclude both atomic and nuclear physics explanations [27]. Elementary particle physics explanations [27, 28] seem to be excluded already by the fact that several peaks have been observed in the range $1.6 - 1.8\ MeV$ with widths of order $10^1 - 10^2\ keV$. These states decay to e^+e^- pairs. There is evidence for one narrow peak with width of order one keV at 1.062 Mev [27]: this state decays to photon-photon pairs.

Thus it seems that the structures produced might be composite, perhaps resonances in e^+e^- system. The difficulty of this explanation is that conventional QED seems to offer no natural explanation for the strong force needed to explain the energy scale of the states. One idea is that the strong electromagnetic fields create a new phase of QED [27] and that the resonances are analogous to pseudoscalar mesons appearing as resonances in strongly interacting systems.

TGD based explanation relies on the following hypothesis motivated by Topological Geometrodynamics.

1. Ordinary leptons are not point like particles and can have colored excitations, which form color singlet bound states. A natural identification for the primary condensate level is $k = 121$ so that the mass scale is of order one MeV for the states containing lowest generation colored leptons. The fact that d and s quarks, having the same weak quantum numbers as charged leptons, have same primary condensation level, suggests that both colored electron and muon condense to the same level. The expectation that lepto-hadron physics exists in a narrow energy interval only, suggests that also colored τ should condense on the same level.

2. The states in question are lepto-hadrons, that is color confined states formed from the colored excitations of e^+ and e^-. The decay rate to lepto-nucleon pairs $e^+_{ex}e^-_{ex}$ is large and turns out to give rise to correct order of magnitude for the decay width. Hence two options emerge.

   Option 1: Lepto-nucleons e_{ex} have masses only slightly above the electron mass and since they behave like electrons, anomalous e^+e^- pairs could actually correspond to lepto-nucleon pairs created in the decays of lepto-pions. 1.062, 1.63 and 1.77 MeV states can be identified as lowest generation lepto-pions correspond to octet and two decuplets. 1.83 MeV state could be identified as the second generation lepto-pion corresponding to colored muon. The small branching fraction to gamma pairs explains why the decays of the higher mass lepto-pions to gamma pairs has not been observed. $g = 0$ lepto-pion decays to lepto-nucleon pairs can be visualized as occuring via dual diagrams obeying Zweig's rule (annihilation is not allowed inside incoming or outgoing particle states). The decay of $g = 1$ colored muon pair occurs via Zweig rule violating annihilation to two gluon intermediate state, which transforms back to virtual $g = 0$ colored electron pair decaying via dual diagram: the violation of Zweig's rule suggests that the decay rate for 1.8 MeV state is smaller than for the lighter states. Quantitive model shows that this scenario is the most plausible one.

Option 2: Lepto-sigmas, which are the scalar partners of lepto-pions predicted by sigma model, are the source of anomalous (and genuine) e^+e^- pairs. In this case 1.062 state must correspond to lepto-pion whereas higher states must be identified as lepto-sigmas. Also now new lepto-pion states decaying to gamma pairs are predicted and one could hence argue that this prediction is not consistent with what has been observed. A crucial assumption is that lepto-sigmas are light and cannot decay to other lepto-mesons. Ordinary hadronic physics suggests that this need not be the case: the hadronic decay width of the ordinary sigma, if it exists, is very large.

The program of the section is following:

1. PCAC hypothesis, successful in low energy pion physics, is generalized to the case of lepto-pion. Hypothesis allows to deduce the coupling of lepto-pion to leptons and lepto-baryons in terms of leptobaryon-lepton mixing angles. Ortopositronium anomaly allows to deduce precise value of $f(\pi_L)$ characterizing the decay rate of lepto-pion so that the crucial parameters of the model are completely fixed. The decay rates of lepto-pion to photon pair and of lepto-sigma to ordinary e^+e^- pairs are within experimental bounds and corrections to muon and beta decay rates are small. New calculable resonance contributions to photon-photon scattering at cm energy equal to lepto-pion masses are predicted.

2. If anomalous e^+e^- pairs are actually lepto-nucleon pairs, only a model for the creation of lepto-pions from vacuum is needed. In an external electromagnetic field lepto-pion develops a vacuum expectation value proportional to electromagnetic anomaly term [29] so that the production amplitude for the lepto-pion is essentially the Fourier transform of the scalar product of the electric field of the stationary target nucleus with the magnetic field of the colliding nucleus.

3. If anomalous e^+e^- pairs are produced in the decays of lepto-sigmas, the starting point is sigma model providing a realization of PCAC hypothesis. Sigma model makes it possible to relate the production amplitude for $\sigma_L\pi_L$ pairs to the lepto-pion production amplitude: the key element of the model is the large value of the $\sigma\pi_L\pi_L$ coupling constant.

4. Lepto-hadron production amplitudes are proportional to lepto-pion production amplitude and this motivates a detailed study of lepto-pion production. Two models for lepto-pion production are developed: in classical model colliding nucleus is treated classically whereas in quantum model the colliding nucleus is described quantum mechanically. It turns out that classical model explains the peculiar production characteristics of lepto-pion but that production cross section is too small by several orders of magnitude. Quantum mechanical model predicts also diffractive effects: production cross section varies rapidly as a function of the scattering angle and for a fixed value of scattering angle there is a rapid variation with the collision velocity. The estimate for the total lepto-pion production cross section increases by several orders of magnitude due to the coherent summation of the contributions to the amplitude from different values of the impact parameter at the peak.

5. The production rate for lepto-nucleon pairs is only slightly smaller than the experimental upper bound but the e^+e- production rate predicted by sigma model approach is still by a

factor of order $1/\sum N_c^2$ smaller than the reported maximum cross section. A possible explanation for this discrepancy is the huge value of the coupling $g(\pi_L, \pi_L, \sigma_L) = g(\sigma_L, \sigma_L, \sigma_L)$ implying that the diagram involving the exchange of virtual sigma can give the dominant contribution to the production cross section of $\sigma_L \pi_L$ pair.

2.2 Lepto-pions and generalized PCAC hypothesis

One can say that the PCAC hypothesis predicts the existence of pions and a connection between the pion nucleon coupling strength and the pion decay rate to leptons. In the following we give the PCAC argument and its generalization and consider various consequences.

2.2.1 PCAC for ordinary pions

The PCAC argument for ordinary pions goes as follows [30]:

1. Consider the contribution of the hadronic axial current to the matrix element describing lepton nucleon scattering (say $N + \nu \rightarrow P + e^-$) by weak interactions. The contribution in question reduces to the well-known current-current form

$$
\begin{aligned}
M &= \frac{G_F}{\sqrt{2}} g_A L_\alpha \langle P|A^\alpha|P\rangle \ , \\
L_\alpha &= \bar{e}\gamma_\alpha(1+\gamma_5)\nu \ , \\
\langle P|A^\alpha|P\rangle &= \bar{P}\gamma^\alpha N \ ,
\end{aligned} \tag{1}
$$

   where $G_F = \frac{\pi\alpha}{2m_W^2 sin^2(\theta_W)} \simeq 10^{-5}/m_p^2$ denotes the dimensional weak interaction coupling strength and g_A is the nucleon axial form factor:$g_A \simeq 1.253$.

2. The matrix element of the hadronic axial current is not divergenceless, due to the nonvanishing nucleon mass,

$$
a_\alpha \langle P|A^\alpha|P\rangle \simeq 2m_p \bar{P}\gamma_5 N \ . \tag{2}
$$

   Here q^α denotes the momentum transfer vector. In order to obtain divergenceless current, one can modify the expression for the matrix element of the axial current

$$
\langle P|A^\alpha|N\rangle \rightarrow \langle P|A^\alpha|N\rangle - q^\alpha 2m_p \bar{P}\gamma_5 N \frac{1}{q^2} \ . \tag{3}
$$

3. The modification introduces a new term to the lepton-hadron scattering amplitude identifiable as an exchange of a massless pseudoscalar particle

$$
\delta T = \frac{G_F g_A}{\sqrt{2}} L_\alpha \frac{2m_p q^\alpha}{q^2} \bar{P}\gamma_5 N \ . \tag{4}
$$

   The amplitude is identifiable as the amplitude describing the exchange of the pion, which gets its mass via the breaking of chiral invariance and one obtains by the straightfowread replacement $q^2 \rightarrow q^2 - m_\pi^2$ the correct form of the amplitude.

4. The nontrivial point is that the interpretations as pion exhange is indeed possible since the amplitude obtained is to a good approximation identical to that obtained from the Feynman diagram describing pion exchange, where the pion nucleon coupling constant and pion decay amplitude appear

$$T_2 = \frac{G}{\sqrt{2}} f_\pi q^\alpha L_\alpha \frac{1}{q^2 - m_\pi^2} g\sqrt{2}\bar{P}\gamma_5 N \ . \quad (5)$$

The condition $\delta T \sim T_2$ gives from Goldberger-Treiman [30]

$$g_A(\simeq 1.25) = \sqrt{2}\frac{f_\pi g}{2m_p}(\simeq 1.3) \ , \quad (6)$$

satisfied in a good accuracy experimentally.

2.2.2 PCAC in leptonic sector

A natural question is why not generalize the previous argument to the leptonic sector and look at what one obtains. The generalization is based on following general picture.

1. There are two levels to be considered: the level of ordinary leptons and the level of leptobaryons of, say type $f_{ABC}\nu_8^A\nu_8^B\bar{L}_{10}^C$, possessing same quantum numbers as leptons. The interaction transforming these states to each other causes in mass eigenstates mixing of leptobaryons with ordinary leptons described by mixing angles. The masses of lepton and corresponding leptobaryon could be quite near to each other and in case of electron this should be the case as it turns out.

2. A counterargument against the applications of PCAC hypothesis at level of ordinary leptons is that baryons and mesons are both bound states of quarks whereas ordinary leptons are not bound states of colored leptons. The divergence of the axial current is however completely independent of the possible internal structure of leptons and microscopic emission mechanism. Ordinary lepton cannot emit lepto-pion directly but must first transform to leptobaryon with same quantum numbers: phenomenologically this process can be described using mixing angle $sin(\theta_B)$. The emission of lepto-pion proceeds as $L \rightarrow B_L : B_L \rightarrow B_L + \pi_L$: $B_L \rightarrow L$, where B_L denotes leptobaryon of type structure $f_{ABC}L_8^A L_8^B \bar{L}_8^C$. The transformation amplitude $L \rightarrow B_L$ is proportional to the mixing angle $sin(\theta_L)$.

Three different PCAC type identities are assumed to hold true:
PCAC1) The vertex for the emission of lepto-pion by ordinary lepton is equivalent with the graph in which lepton L transforms to leptobaryon L^{ex} with same quantum numbers, emits lepto-pion and transforms back to ordinary lepton. The assumption relates the couplings $g(L_1, L_2)$ and $g(L_1^{ex}, L_2^{ex})$ (analogous to strong coupling) and mixing angles to each other

$$g(L_1, L_2) = g(L_1^{ex}, L_2^{ex}) sin(\theta_1) sin(\theta_2) \ . \quad (7)$$

The condition implies that in electro-weak interactions ordinary leptons do not transform to their exotic counterparts.
PCAC2) The generalization of the ordinary Goldberger-Treiman argument holds true, when ordinary baryons are replaced with leptobaryons. This gives the condition expressing the coupling $f(\pi_L)$ of the lepto-pion state to axial current defined as

$$\langle vac|A_\alpha|\pi_L\rangle = ip_\alpha f(\pi_L) \ , \tag{8}$$

in terms of the masses of leptobaryons and strong coupling g.

$$f(\pi_L) \ = \ \sqrt{2}g_A\frac{(m_{ex}(1)+m_{ex}(2))sin(\theta_1)sin(\theta_2)}{g(L_1,L_2)} \ , \tag{9}$$

where g_A is parameter characterizing the deviation of weak coupling strength associated with leptobaryon from ideal value: $g_A \sim 1$ holds true in good approximation.
PCAC3) The elimination of leptonic axial anomaly from leptonic current fixes the values of $g(L_i,L_j)$.

i) The standard contribution to the scattering of leptons by weak interactions given by the expression

$$\begin{aligned} T &= \frac{G_F}{\sqrt{2}}\langle L_1|A^\alpha|L_2\rangle\langle L_3|A_\alpha|L_4\rangle \ , \\ \langle L_i|A^\alpha|L_j\rangle &= \bar{L}_i\gamma^\alpha\gamma_5 L_j \ . \end{aligned} \tag{10}$$

ii) The elimination of the leptonic axial anomaly

$$q_\alpha\langle L_i|A^\alpha|L_j\rangle \ = \ (m(L_i)+m(L_j))\bar{L}_i\gamma_5 L_j \ , \tag{11}$$

by modifying the axial current by the anomaly term

$$\langle L_i|A^\alpha|L_j\rangle \ \rightarrow \ \langle L_i|A^\alpha|L_j\rangle - (m(L_i)+m(L_j))\frac{q^\alpha}{q^2}\bar{L}_i\gamma_5 L_j \ , \tag{12}$$

induces a new interaction term in the scattering of ordinary leptons.

iii) It is assumed that this term is equivalent with the exchange of lepto-pion. This fixes the value of the coupling constant $g(L_1,L_2)$ to

$$\begin{aligned} g(L_1,L_2) &= 2^{1/4}\sqrt{G_F}(m(L_1)+m(L_2))\xi \ , \\ \xi(charged) &= 1 \ , \\ \xi(neutral) &= cos(\theta_W) \ . \end{aligned} \tag{13}$$

Here the coefficient ξ is related to different values of masses for gauge bosons W and Z appearing in charged and neutral current interactions. An important factor 2 comes from the modification of the axial current in both matrix elements of the axial current.

Lepto-pion exchange interaction couples right and left handed leptons to each other and its strength is of the same order of magnitude as the strength of the ordinary weak interaction at energies not considerably large than the mass of the lepto-pion. At high energies this interaction is negligible and the existence of the lepto-pion predicts no corrections to the parameters of the standard model since these are determined from weak interactions at much higher energies. If lepto-pion mass is sufficiently small (as found, $m(\pi_L) < 2m_e$ is allowed by the experimental data), the interaction mediated by lepto-pion exchange can become quite strong due to the presence of the lepto-pion progator. The value of the lepton-lepto-pion coupling is $g(e,e) \equiv g \sim 5.6\cdot 10^{-6}$. It is perhaps worth noticing that the value of the coupling constant is of the same order as lepton-Higgs coupling constant and also proportional to the mass of the lepton.

PCAC identities fix the values of coupling constants apart from the values of mixing angles. If one assumes that the strong interaction mediated by lepto-pions is really strong and the coupling strength $g(L_{ex}, L_{ex})$ is of same order of magnitude as the ordinary pion nucleon coupling strength $g(\pi NN) \simeq 13.5$ one obtains an estimate for the value of the mixing angle $sin(\theta_e)$ $sin^2(\theta_e) \sim \frac{g(\pi NN)}{g(L,L)} \sim 2.4 \cdot 10^{-6}$. This implies the order of magnitude $f(\pi_L) \sim 10^{-6} m_W \sim 10^2\ keV$ for $f(\pi_L)$. The order of magnitude is correct as will be found. Ortopositronium decay rate anomaly $\Delta\Gamma/\Gamma \sim 10^{-3}$ and the assumption $m_{ex} \geq 1.3\ MeV$ (so that $e_{ex}\bar{e}$ decay is not possible) gives the upper bound $sin(\theta_e) \leq x \cdot \sqrt{N_c} \cdot 10^{-4}$, where the value of $x \sim 1$ depends on the number of the lepto-pion type states and on the precise value of the Op anomaly.

2.3 Lepto-pion decays and PCAC hypothesis

The PCAC argument makes it possible to predict the lepto-pion coupling and decay rates of the lepto-pion to various channels. Actually the orders of magnitude for the decay rates of the lepto-sigma and other lepto-mesons can be deduced also. The comparison with the experimental data is made difficult by the uncertainty of the identifications. The lightest candidate has mass $1.062\ MeV$ and decay width of order 1 keV [27]: only photon photon decay has been observed for this state. The next lepto-meson candidates are in the mass range $1.6 - 1.8\ MeV$. Perhaps the best status is possessed by 'Darmstadtium' with mass $1.8\ MeV$. For these states decays to final states identified as e^+e^- pairs dominate: if indeed e^+e^- pairs, these states probably correspond to the decay products of lepto-sigma. Another possibility is that pairs are actually lepto-nucleon pairs with the mass of the lepto-nucleon only slightly larger than electron mass. Hadron physics experience suggests that the decay widths of the lepto-hadrons (lepto-pion forming a possible exception) should be about 1-10 per cent of particle mass as in hadron physics. The upper bounds for the widths are indeed in the range $50 - 70\ keV$ [27].

2.3.1 $\Gamma(\pi_L \rightarrow \gamma\gamma)$

As in the case of the ordinary pion, anomaly considerations give the following approximate expression for the decay rate of the lepto-pion to two-photon final states [29])

$$\Gamma(\pi_L \rightarrow \gamma\gamma) = \frac{N_c^2 \alpha^2 m^3(\pi_L)}{64 f(\pi_L)^2 \pi^3} . \tag{14}$$

$N_c = 8, 10$ is the number of the colored lepton states coming from the axial anomaly loop. For $m(\pi_L) = 1.062\ MeV$ and $f(\pi_L) = N_c \cdot 7.9\ keV$ implied by the ortopositronium decay rate anomaly $\Delta\Gamma/\Gamma = 10^{-3}$ one has $\Gamma(\gamma\gamma) = .52\ keV$, which is consistent with the experimental estimate of order $1\ keV$ [27].

In fact, several lepto-pion states could exist (4 at least corresponding to the resonances at 1.062, 1.63, 1.77 and 1.83 MeV). Since all these lepto-pion states contribute to Op decay rate, the actual value of $f(\pi_L)$ assumed to scale as $m(\pi_L)$, is actually larger in this case: it turns out that $f(\pi_L)$ for the lightest lepto-pion increases to $f(\pi_L)(lightest) = N_c \cdot 15\ keV$ and gives $\Gamma(\gamma\gamma) \simeq .13\ keV$ in case of the lightest lepto-pion if lepto-pions are assumed to correspond the resonances. Note that the order of magnitude for $f(\pi_L)$ is same as deduced from the assumption that lepto-hadronic counterpart of $g(\pi NN)$ equals to the ordinary $g(\pi NN)$. The increase of the ortopositronium anomaly by a factor of, say 4, implies corresponding decrease in $f(\pi_L)^2$. The value of $f(\pi_L)$ is also sensitive to the precise value of the mass of the lightest lepto-pion.

2.3.2 Lepto-pion-lepton coupling

The value of the lepto-pion-lepton coupling can be used to predict the decay rate of lepto-pion to leptons. One obtains for the decay rate $\pi_L^0 \rightarrow e^+e^-$ the estimate

$$
\begin{aligned}
\Gamma(\pi_L \rightarrow e^+e^-) &= 4\frac{g(e,e)^2\pi}{2(2\pi)^2}(1-4x^2)m(\pi_L) \\
&= 16Gm_e^2 cos^2(\theta_W)\frac{\sqrt{2}}{4\pi}(1-4x^2)m(\pi_L) \ , \\
x &= \frac{m_e}{m(\pi_L)} \ . \qquad (15)
\end{aligned}
$$

for the decay rate of the lepto-pion: for lepto-pion mass $m(\pi_L) \simeq 1.062\ MeV$ one obtains for the decay rate the estimate $\Gamma \sim 1/(1.3 \cdot 10^{-8}\ sec)$: the low decay rate is partly due to the phase space suppression and implies that e^+e^- decay products cannot be observed in the measurement volume. The low decay rate is in accordance with the identification of the lepto-pion as the $m = 1.062\ MeV$ lepto-pion candidate. In sigma model lepto-pion and lepto-sigma have identical lifetimes and for lepto-sigma mass of order 1.8 MeV one obtains $\Gamma(\sigma_L \rightarrow e^+e^-) \simeq 1/(8.2 \cdot 10^{-10}\ sec)$: the prediction is larger than the lower limit $\sim 1/(10^{-9}\ sec)$ for the decay rate implied by the requirement that σ_L decays inside the measurement volume. The estimates of the lifetime obtained from heavy ion collisions [31] give the estimate $\tau \geq 10^{-10}\ sec$. The large value of the lifetime is in accordance with the limits for the lifetime obtained from Babbha scattering [32], which indicate that the lifetime must be longer than 10^{-12} sec.

For lepto-meson candidates with mass above 1.6 MeV no experimental evidence for other decay modes than $X \rightarrow e^+e^-$ has been found and the empirical upper limit for $\gamma\gamma/e^+e^-$ branching ratio [33] is $\Gamma(\gamma\gamma)/\Gamma(e^+e^-) \leq 10^{-3}$. If the identification of the decay products as e^+e^- pairs is correct then the only possible conclusion is that these states cannot correspond to lepto-pion since lepto-pion should decay dominantly into photon photon pairs. Situation changes if pairs of lepton-ucleons $e_{ex}\bar{e}_{ex}$ of type $e_{ex} = e_8\nu_8\overline{\nu}_8$ pair are in question.

I realized that this conclusion might be questioned for more than decade after writing the above text as I developed a model for CDF anomaly suggesting the existence of τ-pions. Since colored leptons are color octets, anomalous magnetic moment type coupling of form $\overline{L}Tr(F^{\mu\nu}\Sigma_{\mu\nu}L_8)$ (the trace is over the Lie-algebra generators of $SU(3)$ and $F^{\mu\nu}$ denotes color gauge field) between ordinary lepton, colored lepton and lepto-gluon is possible. The exchange of a virtual lepto-gluon allows lepto-pion to decay by lepto-strong interactions to electron-positron pairs. The decay rate is limited by the kinematics for the lightest state very near to the final state mass and might make decay rate to in this case very small. If the rate for the decay to electron-positron pair is comparable to that for the decay to two photons the production rate for electron-positron pairs could be of the same order of magnitude as leptopion production rate. The anomalous magnetic moment of electron however poses strong limitations on this coupling and it might be that the coupling is too small. This coupling could however induce the mixing of e_{ex} with e.

2.3.3 $\Gamma(\pi_L \rightarrow e + \bar{\nu_e})$

The expression for the decay rate $\pi_L \rightarrow e + \bar{\nu_e}$ reads as

$$
\begin{aligned}
\Gamma(\pi_L^- \rightarrow e\nu_e) &= 8Gm_e^2\frac{(1-x^2)^2}{2(1+x^2)}\frac{\sqrt{2}}{(2\pi)^5}m(\pi_L) \ , \\
&= \frac{4}{cos^2(\theta_W)}\frac{(1-x^2)}{(1+x^2)(1-4x^2)}\Gamma(\pi_L^0 \rightarrow e^+e^-) \ , \qquad (16)
\end{aligned}
$$

and gives $\Gamma(\pi_L^- \rightarrow e\nu_e) \simeq 1/(3.6 \cdot 10^{-10}\ sec)$ for $m(\pi_L) = 1.062\ MeV$.

2.3.4 $\Gamma(\pi_L/\sigma_L \rightarrow e_{ex}\bar{e}_{ex})$ and $\Gamma(\pi_L/\sigma_L \rightarrow e_{ex}\bar{e})$

Sigma model predicts lepto-pion and lepto-sigma to have same coupling to lepto-nucleon e_{ex} pair so that in the sequel only lepto-pion decay rates are considered. One must consider also the possibility that lepto-pion decay products are either $e_{ex}\bar{e}_{ex}$ or $e_{ex}\bar{e}$ pairs with e_{ex} having mass of near the mass of electron so that it could be misidentified as electron. If the mass of lepto-nucleon e_{ex} with quantum numbers of electron is smaller than $m(\pi_L)/2$ it can be produced in lepto-pion annihilation. One must also assume $m(e_{ex}) > m_e$: otherwise electrons would spontaneously decay to lepto-nucleons via photon emission. The production rate to lepto-nucleon pair can be written as

$$
\begin{aligned}
\Gamma(\pi_L \rightarrow e_{ex}^+ e_{ex}^-) &= \frac{1}{sin^4(\theta_e)} \frac{(1-4y^2)}{(1-4x^2)} \Gamma(\pi_L \rightarrow e^+e^-) \ , \\
y &= \frac{m(e_{ex})}{m(\pi_L)} \ .
\end{aligned} \tag{17}
$$

If $e - e_{ex}$ mass difference is sufficiently small the kinematic suppression does not differ significantly from that for e^+e^- pair. The limits from Babbha scattering give no bounds on the rate of $\pi_L \rightarrow e_{ex}^+ e_{ex}^-$ decay. The decay rate $\Gamma \sim 10^{20}/sec$ implied by $sin(\theta_e) \sim 10^{-4}$ implies decay width of order .1 MeV: the order of magnitude is the naively expected one and means that the decay to $e_{ex}^+ e_{ex}^-$ pairs dominates over the decay to gamma pairs except in the case of the lightest lepto-pion state for which the decay is kinematically forbidden.

The decay rate of the lepto-pion to $\bar{e}e_{ex}$ pair has sensible order of magnitude: for $sin(\theta_e) = 1.2 \cdot 10^{-3}$, $m_{\sigma_L} = 1.8\ MeV$ and $m_{e_{ex}} = 1.3\ MeV$ one has $\Gamma \simeq 60\ eV$ allowed by the experimental limits. This decay is kinematically possible only provided the mass of e_{ex} is in below $1.3\ MeV$. These decays should dominate by a factor $1/sin^2(\theta_e)$ over e^+e^- decays if kinematically allowed.

A signature of these events, if identified erratically as electron positron pairs, is the non-vanishing value of the energy difference in the cm frame of the pair: $E(e^-) - E(e^+) \simeq (m^2(e_{ex}) - m_e^2)/2E > 160\ keV$ for $E = 1.8\ MeV$. If the decay $e_{ex} \rightarrow e + \gamma$ takes place before the detection the energy asymmetry changes its sign. Energy asymmetry [34] increasing with the rest energy of the decaying object has indeed been observed: the proposed interpretation has been that electron forms a bound state with the second nucleus so that its energy is lowered. Also a deviation from the momentum distribution implied by the decay of neutral particle to e^+e^- pair (momenta are opposite in the rest frame) results from the emission of photon. This kind of deviation has also been observed [34]: the proposed explanation is that third object is involved in the decay. A possible alternative explanation for the asymmetries is the production mechanism ($\sigma_L\pi_L$ pairs instead of single particle states).

2.3.5 $\Gamma(e_{ex} \rightarrow e + \gamma)$

The decay to electron and photon would be a unique signature of e_{ex}. The general feature of of fermion family mixing is that mixing takes place in charged currents. In present case mixing is of different type so that $e_{ex} \rightarrow e + \gamma$ might be allowed. If this is not the case then the decay takes place as weak decay via the emission of virtual W boson: $e_{ex} \rightarrow e + \nu_e + \bar{\nu}_e$ and is very slow due to the presence of mixing angle and kinematical supression. The energy of the emitted photon is $E_\gamma = (m_{ex}^2 - m_e^2)/2m_e$. The decay rate $\Gamma(e_{ex} \rightarrow e + \gamma)$ is given by

$$
\Gamma(e_{ex} \rightarrow e + \gamma) = \alpha_{em} sin^2(\theta_e) X m_e \ ,
$$

$$X = \frac{(m_1 - m_e)^3(m_1 + m_e)m_e}{(m_1^2 + m_e^2)^2 m_1} \quad . \tag{18}$$

For $m(e_{ex}) = 1.3\ MeV$ the decay of order $1/(1.4 \cdot 10^{-12}\ sec)$ for $sin(\theta_e) = 1.2 \cdot 10^{-3}$ so that lepto-nucleons would decay to electrons in the measurement volume. In the experiments positrons are identified via pair annihilation and since pair annihilation rate for $\bar{e}_{ex}$ is by a factor $sin^2(\theta_e)$ slower than for e^+ the particles identified as positrons must indeed be positrons. For sufficiently small mass difference $m(e_{ex}) - m_e$ the particles identified as electron could actually be e_{ex}. The decay of e_{ex} to electron plus photon before its detection seems however more reasonable alternative since it could explain the observed energy asymmetry [34].

2.3.6 Some implications

The results have several implications as far as the decays of on mass shell states are considered:

1. For $m(e_{ex}) > 1.3\ MeV$ the only kinematically possible decay mode is the decay to e^+e^- pair. Production mechanism might explain the asymmetries [34]. The decay rate of on mass shell π_L and σ_L (or $\eta_L, \rho_L, ..$) is above the lower limit allowed by the detection in the measurement volume.

2. If the mass of e_{ex} is larger than $.9\ MeV$ but smaller than $1.3\ MeV$ $e_{ex}\bar{e}$ decays dominate over e^+e^- decays. The decay $e_{ex} \to e + \gamma$ before detection could explain the observed energy asymmetry.

3. It will be found that the direct production of $e_{ex}\bar{e}$ pairs is also possible in the heavy ion collision but the rate is much smaller due to the smaller phase space volume in two-particle case. The annihilation rate of $\bar{e}_{ex}$ in matter is by a factor $sin^2(\theta_e)$ smaller than the annihilation rate of positron. This provides a unique signature of e_{ex} if e^+ annihilation rate in matter is larger than the decay rate of $\bar{e}_{ex}$. In lead the lifetime of positron is $\tau \sim 10^{-10}\ sec$ and indeed larger that e_{ex} lifetime.

2.3.7 Karmen anomaly

A brief comment on the Karmen anomaly [18] observed in the decays of π^+ is in order. The anomaly suggests the existence [19] of new weakly interacting neutral particle x, which mixes with muon neutrino. Since $g = 1$ neutrino corresponds to charmed quark in hadron physics context having $k = 103$ rather than $k = 107$ as primary condensation level, a natural guess for its primary condensation level is $k = 113$, which would mean that the mass scale would be of order muon mass: the particle candidate indeed has mass of order 30 MeV. One class of solutions to laboratory constraints, which might evade also cosmological and astrophysical constraints, corresponds to object x mixing with muon type neutrino and decaying radiatively to $\gamma + \nu_\mu$ via the emission of virtual W boson. The value of the mixing parameter $U(\mu, x)$ describing $\nu_{mu} - x$ mixing satisfies $|U_{\mu,x}|^4 \simeq .8 \cdot 10^{-10}$.

The following naive PCAC argument gives order of magnitude estimate for $|U(\mu, x)| \sim sin(\theta_\mu)$. The value of $g(\mu, \mu)$ is by a factor $m(\mu)/m_e$ larger than $g(e, e)$. If the lepto-hadronic couplings $g(\mu_{ex}, \mu_{ex})$ and $g(e_{ex}, e_{ex})$ are of same order of magnitude then one has $sin(\theta_\mu) \leq .02$ (3 lepto-pion states and Op anomaly equal to $Op = 5 \cdot 10^{-3}$): the lower bound is 6.5 times larger than the value .003 deduced in [19]. The actual value could be considerably smaller since e_{ex} mass could be larger than 1.3 MeV by a factor of order 10.

2.4 Lepto-pions and weak decays

The couplings of lepto-meson to electro-weak gauge bosons can be estimated using PCAC and CVC hypothesis [29]. The effective $m_{\pi_L} - W$ vertex is the matrix element of electro-weak axial current between vacuum and charged lepto-meson state and can be deduced using same arguments as in the case of ordinary charged pion

$$\langle 0|J_A^\alpha|\pi_L^-\rangle = Km(\pi_l)p^\alpha \quad , \tag{19}$$

where K is some numerical factor and p^α denotes the momentum of lepto-pion. For neutral lepto-pion the same argument gives vanishing coupling to photon by the conservation of vector current. This has the important consequence that lepto-pion cannot be observed as resonance in e^+e^- annihilation in single photon channel. In two photon channel lepto-pion should appear as resonance. The effective interaction Lagrangian is the 'instanton' density of electromagnetic field giving additional contribution to the divergence of the axial current and was used to derive a model for lepto-pion production in heavy ion collisions.

2.4.1 Lepto-hadrons and lepton decays

The lifetime of charged lepto-pion is from PCAC estimates larger than 10^{-10} seconds by the previous PCAC estimates. Therefore lepto-pions are practically stable particles and can appear in the final states of particle reactions. In particular, lepto-pion atoms are possible and by Bose statistics have the peculiar property that ground state can contain many lepto-pions.

Lepton decays $L \to \nu_\mu + H_L$, $L = e, \mu, \tau$ via emission of virtual W are kinematically allowed and an anomalous resonance peak in the neutrino energy spectrum at energy

$$E(\nu_L) = \frac{m(L)}{2} - \frac{m_H^2}{2m(L)} \quad , \tag{20}$$

provides a unique test for the lepto-hadron hypothesis. If lepto-pion is too light electrons would decay to charged lepto-pions and neutrinos unless the condition $m(\pi_L) > m_e$ holds true.

The existence of a new decay channel for muon is an obvious danger to the lepto-hadron scenario: large changes in muon decay rate are not allowed.

Consider first the decay $\mu \to \nu_\mu + \pi_L$ where π_L is on mass shell lepto-pion. Lepto-pion has energy $\sim m(\mu)/2$ in muon rest system and is highly relativistic so that in the muon rest system the lifetime of lepto-pion is of order $\frac{m(\mu)}{2m(\pi_L)}\tau(\pi_L)$ and the average length traveled by lepto-pion before decay is of order 10^8 meters! This means that lepto-pion can be treated as stable particle. The presence of a new decay channel changes the lifetime of muon although the rate for events using $e\nu_e$ pair as signature is not changed. The effective $H_L - W$ vertex was deduced above. The rate for the decay via lepto-pion emission and its ratio to ordinary rate for muon decay are given by

$$\begin{aligned} \Gamma(\mu \to \nu_\mu + H_L) &= \frac{G^2K^2}{2^5\pi} m^4(\mu)m^2(H_L)(1 - \frac{m^2(H_L)}{m^2(\mu)})\frac{(m^2(\mu) - m^2(H_L))}{(m^2(\mu) + m^2(H_L))} \quad , \\ \frac{\Gamma(\mu \to \nu_\mu + H_L)}{\Gamma(\mu \to \nu_\mu + e + \bar{\nu}_e)} &= 6 \cdot (2\pi^4)K^2 \frac{m^2(H_L)}{m^2(\mu)} \frac{(m^2(\mu) - m^2(H_L))}{(m^2(\mu) + m^2(H_L))} \quad , \end{aligned} \tag{21}$$

and is of order $.93K^2$ in case of lepto-pion. As far as the determination of G_F or equivalently m_W^2 from muon decay rate is considered the situation seems to be good since the change introduced to G_F is of order $\Delta G_F/G_F \simeq 0.93K^2$ so that K must be considerably smaller than one. For the physical value of K: $K \leq 10^{-2}$ the contribution to the muon decay rate is negligible.

Lepto-hadrons can appear also as virtual particles in the decay amplitude $\mu \rightarrow \nu_\mu + e\nu_e$ and this changes the value of muon decay rate. The correction is however extremely small since the decay vertex of intermediate off mass shell lepto-pion is proportional to its decay rate.

2.4.2 Lepto-pions and beta decay

If lepto-pions are allowed as final state particles lepto-pion emission provides a new channel $n \rightarrow p + \pi_L$ for beta decay of nuclei since the invariant mass of virtual W boson varies within the range $(m_e = 0.511\ MeV, m_n - m_p = 1.293 MeV$. The resonance peak for $m(\pi_L) \simeq 1\ MeV$ is extremely sharp due to the long lifetime of the charged lepto-pion. The energy of the lepto-pion at resonance is

$$E(\pi_L) = (m_n - m_p)\frac{(m_n + m_p)}{2m_n} + \frac{m(\pi_L)^2}{2m_n} \simeq m_n - m_p \ . \tag{22}$$

Together with long lifetime this lepto-pions escape the detector volume without decaying (the exact knowledge of the energy of charged lepto-pion might make possible its direct detection).

The contribution of lepto-pion to neutron decay rate is not negligible. Decay amplitude is proportional to superposition of divergences of axial and vector currents between proton and neutron states.

$$M = \frac{G}{\sqrt{2}} K m(\pi_L)(q^\alpha V_\alpha + q^\alpha A_\alpha) \ . \tag{23}$$

For exactly conserved vector current the contribution of vector current vanishes identically. The matrix element of the divergence of axial vector current at small momentum transfer (approximately zero) is in good approximation given by

$$\begin{aligned} \langle p|q^\alpha A_\alpha|n\rangle &= g_A(m_p + m_n)\bar{u}_p\gamma_5 u_n \ , \\ g_A &\simeq 1.253 \ . \end{aligned} \tag{24}$$

Straightforward calculation shows that the ratio for the decay rate via lepto-pion emission and ordinary beta decay rate is in good approximation given by

$$\begin{aligned} \frac{\Gamma(n \rightarrow p + \pi_L)}{\Gamma(n \rightarrow p + e + \bar{\nu}_e)} &= \frac{30\pi^2 g_A^2 K^2}{0.47 \cdot (1 + 3g_A^2)} \frac{m_{\pi_L}^2 (\Delta^2 - m_{\pi_L}^2)^2}{\Delta^6} \ , \\ \Delta &= m(n) - m(p) \ . \end{aligned} \tag{25}$$

Lepto-pion contribution is smaller than ordinary contribution if the condition

$$K < \left[\frac{.47 \cdot (1 + 3g_A^2)}{30\pi^2 g_A^2} \frac{\Delta^6}{(\Delta^2 - m_{\pi_L}^2)^2 m_{\pi_L}^2}\right]^{1/2} \simeq .28 \ , \tag{26}$$

is satisfied. The upper bound $K \leq 10^{-2}$ coming from the lepto-pion decay width and Op anomaly implies that the contribution of the lepto-pion to beta decay rate is very small.

2.5 Ortopositronium puzzle and lepto-pion in photon photon scattering

The decay rate of ortopositronium (Op) has been found to be slightly larger than the rate predicted by QED [20, 35]: the discrepancy is of order $\Delta\Gamma/\Gamma \sim 10^{-3}$. For parapositronium no anomaly has been observed. Most of the proposed explanations [35] are based on the decay mode $Op \rightarrow X + \gamma$, where X is some exotic particle. The experimental limits on the branching ratio $\Gamma(Op \rightarrow X + \gamma)$ are below the required value of order 10^{-3}. This explanation is excluded also by the standard cosmology [35].

Lepto-pion hypothesis suggests an obvious solution of the Op-puzzle. The increase in annihilation rate is due to the additional contribution to $Op \rightarrow 3\gamma$ decay coming from the decay $Op \rightarrow \gamma_V$ (V denotes 'virtual') followed by the decay $\gamma_V \rightarrow \gamma + \pi_L^V$ followed by the decay $\pi_L^V \rightarrow \gamma + \gamma$ of the virtual lepto-pion to two photon state. $\gamma\gamma\pi_L$ vertices are induced by the axial current anomaly $\propto E \cdot B$. Also a modification of parapositronium decay rate is predicted. The first contribution comes from the decay $Op \rightarrow \pi_L^V \rightarrow \gamma + \gamma$ but the contribution is very small due the smallness of the coupling $g(e, e)$. The second contribution obtained from ortopositronium contribution by replacing one outgoing photon with a loop photon is also small. Since the production of a real lepto-pion is impossible, the mechanism is consistent with the experimental constraints.

The modification to the Op annihilation amplitude comes in a good approximation from the interference term between the ordinary e^+e^- annihilation amplitude F_{st} and lepto-pion induced annihilation amplitude F_{new}:

$$\Delta\Gamma \quad \propto \quad 2Re(F_{st}\bar{F}_{new}) \ , \tag{27}$$

and rough order of magnitude estimate suggests $\Delta\Gamma/\Gamma \sim K^2/e^2 = \alpha^2/4\pi \sim 10^{-3}$. It turns out that the sign and the order of magnitude of the new contribution are correct for $f(\pi_L) \sim 2\ keV$ deduced also from the anomalous e^+e^- production rate.

The new contribution to $e^+e^- \rightarrow 3\gamma$ decay amplitude is most easily derivable using for lepto-pion-photon interaction the effective action

$$\begin{aligned} L_1 &= K\pi_L F \wedge F \ , \\ K &= \frac{\alpha_{em}N_c}{8\pi f(\pi_L)} \ , \end{aligned} \tag{28}$$

where F is quantized electromagnetic field. The calculation of the lepto-pion contribution proceeds in manner described in [29], where the expression for the standard contribution and an elegant method for treating the average over e^+e^- spin triplet states and sum over photon polarizations, can be found. The contribution to the decay rate can be written as

$$\begin{aligned} \frac{\Delta\Gamma}{\Gamma} &\simeq K_1 I_0 \ , \\ K_1 &= \frac{3\alpha N_c^2}{(\pi^2-9)2^9(2\pi)^3}(\frac{m_e}{f(\pi_L)})^2 \ , \\ I_0 &= \int_0^1 \int_{-1}^{umax} \frac{f}{v+f-1-x^2} v^2(2(f-v)u + 2 - v - f) dv du \ , \\ f &\equiv f(v,u) = 1 - \frac{v}{2} - \sqrt{(1-\frac{v}{2})^2 - \frac{1-v}{1-u}} \ , \\ u &= \bar{n}_1 \cdot \bar{n}_2 \ , \ \bar{n}_i = \frac{\bar{k}_i}{\omega_i} \ , \ umax = \frac{(\frac{v}{2})^2}{(1-\frac{v}{2})^2} \ , \\ v &= \frac{\omega_3}{m_e} \ , \ x = \frac{m_{\pi_L}}{2m_e} \ . \end{aligned} \tag{29}$$

ω_i and $\bar{k}_i$ denote the energies of photons, u denotes the cosine of the angle between first and second photon and v is the energy of the third photon using electron mass as unit. The condition $\Delta\Gamma/\Gamma = 10^{-3}$ gives for the parameter $f(\pi_L)$ the value $f(\pi_L)(1.062\ MeV) \simeq N_c \cdot 7.9\ keV$. If there are several lepto-pion states, they contribute to the decay anomaly additively. If the four known resonances correspond directly to lepto-pions decaying to lepto-nucleon pairs and $f(\pi_L)$ is assumed to scale as $N_c m_{\pi_L}$, one obtains $f(\pi_L)(1.062\ MeV) \simeq N_c \cdot 14.7\ keV$. From the PCAC relation one obtains for $sin(\theta_e)$ the upper bound $sin(\theta_e) \leq x \cdot \sqrt{N_c} 10^{-4}$ assuming $m_{ex} \geq 1.3\ MeV$ (so that $e_{ex}\bar{e}$ decay is not possible), where $x = 1.2$ for single lepto-pion state and $x = 1.36$ for four lepto-pion states identified as the observed resonances.

Lepto-pion photon interaction implies also a new contribution to photon-photon scattering. Just at the threshold $E = m_{\pi_L}/2$ the creation of lepto-pion in photon photon scattering is possible and the appearance of lepto-pion as virtual particle gives resonance type behaviour to photon photon scattering near $s = m^2_{\pi_L}$. The total photon-photon cross section in zero decay width approximation is given by

$$\sigma = \frac{\alpha^4 N_c^2}{2^{14}(2\pi)^6} \frac{E^6}{f^4_{\pi_L}(E^2 - \frac{m^2_{\pi_L}}{4})^2} . \tag{30}$$

N	$Op/10^{-3}$	$f(\pi_L)/(N_c keV)$	$sin(\theta_e)(m_{ex}/1.3\ MeV)^{1/2}$	$\Gamma(\pi_L)/keV$
1	1	7.9	$1.2 \cdot 10^{-4}\sqrt{N_c}$	.51
3	1	14.7	$1.7 \cdot 10^{-4}\sqrt{N_c}$	.13
3	5	6.5	$3.6 \cdot 10^{-4}\sqrt{N_c}$	.73

Table 1: The dependence of various quantities on the number of lepto-pion type states and Op anomaly, whose value is varied assuming the proportionality $f(\pi_L) \propto N_c m_{\pi_L}$. N_c refers to the number of lepto-pion states in given representation and Op denotes lepto-pion anomaly.

2.6 Spontaneous vacuum expectation of lepto-pion field as source of lepto-pions

The basic assumption in the model of lepto-pion and lepto-hadron production is the spontaneous generation of lepto-pion vacuum expectation value in strong nonorthogonal electric and magnetic fields. This assumption is in fact very natural in TGD [1].

1. The well known relation [29] expressing pion field as a sum of the divergence of axial vector current and anomaly term generalizes to the case of lepto-pion

$$\pi_L = \frac{1}{f(\pi_L) m^2(\pi_L)} (\nabla \cdot j^A + \frac{\alpha_{em} N_c}{2\pi} E \cdot B) . \tag{31}$$

In the case of lepto-pion case the value of $f(\pi_L)$ has been already deduced from PCAC argument. Anomaly term gives rise to pion decay to two photons so that one obtains an estimate for the lifetime of the lepto-pion.

This relation is taken as the basis for the model describing also the production of lepto-pion in external electromagnetic field. The idea is that the presence of external electromagnetic

[1]'Instanton density' generates coherent state of lepto-pions just like classical em current generates coherent state of photons

field gives rise to a vacuum expectation value of lepto-pion field. Vacuum expectation is obtained by assuming that the vacuum expectation value of axial vector current vanishes.

$$\begin{aligned}
\langle vac \mid \pi \mid vac \rangle &= KE\cdot B \ , \\
K &= \frac{\alpha_{em}N_c}{2\pi f(\pi_L)m^2(\pi_L)} \ .
\end{aligned} \tag{32}$$

Some comments concerning this hypothesis are in order here:

i) The basic hypothesis making possible to avoid large parity breaking effects in atomic and molecular physics is that p-adic condensation levels with length scale $L(n) < 10^{-6}\ m$ are purely electromagnetic in the sense that nuclei feed their Z^0 charges on condensate levels with $L(n) \geq 10^{-6}\ m$. The absence of Z^0 charges does not however exclude the possibility of the classical Z^0 fields induced by the nonorthogonality of the ordinary electric and magnetic fields (if Z^0 fields vanish E and B are orthogonal in TGD.

ii) The nonvanishing vacuum expectation value of the lepto-pion field implies parity breaking in atomic length scales. This is understandable from basic principles of TGD since classical Z^0 field has parity breaking axial coupling to electrons and protons. The nonvanishing classical lepto-pion field is in fact more or less equivalent with the presence of classical Z^0 field.

2. The amplitude for the production of lepto-pion with four momentum $p = (p_0, \bar{p})$ in an external electromagnetic field can be deduced by writing lepto-pion field as sum of classical and quantum parts: $\pi_L = \pi_L(class) + \pi_L(quant)$ and by decomposing the mass term into interaction term plus c-number term and standard mass term:

$$\begin{aligned}
\frac{m^2(\pi_L)\pi_L^2}{2} &= L_{int} + L_0 \ , \\
L_0 &= \frac{m^2(\pi_L)}{2}(\pi_L^2(class) + \pi_L^2(quant)) \ , \\
L_{int} &= m^2(\pi_L)\pi_L(class)\pi_L(quant) \ .
\end{aligned} \tag{33}$$

Interaction Lagrangian corresponds to L_{int} linear in lepto-pion oscillator operators. Using standard LSZ reduction formula and normalization conventions of [29] one obtains for the probability amplitude for creating lepto-pion of momentum p from vacuum the expression

$$\begin{aligned}
A(p) &\equiv \langle a(p)\pi_L \rangle = (2\pi)^3 m^2(\pi_L) \int f_p(x) \langle vac \mid \pi \mid vac \rangle d^4x \ , \\
f_p &= e^{ip\cdot x} \ .
\end{aligned} \tag{34}$$

The probability for the production of lepto-pion in phase space volume element $d3p$ is obtained by multiplying with the density of states factor $d^3n = V\frac{d^3p}{(2\pi)^3}$:

$$\begin{aligned}
dP &= A|U|^2 V d^3p \ , \\
A &= (\frac{\alpha_{em}N_c^2 m^2(\pi_L)}{2\pi f(\pi_L)})^2 \ , \\
U &= \int e^{ip\cdot x} E\cdot B d^4x \ .
\end{aligned} \tag{35}$$

The first conclusion that one can draw is that nonstatic electromagnetic fields are required for lepto-pion creation since in static fields energy conservation forces lepto-pion to have zero energy and thus prohibits real lepto-pion production. In particular, the spontaneous creation lepto-pion in static Coulombic and magnetic dipole fields of nucleus is impossible.

2.7 Sigma model and creation of lepto-hadrons in electromagnetic fields

2.7.1 Why sigma model approach?

For several reasons it is necessary to generalize the model for lepto-pion production to a model for lepto-hadron production.

1. Sigma model approach is necessary if one assumes that anomalous e^+e^- pairs are genuine e^+e^- pairs rather lepto-nucleon pairs produced in the decays of lepto-sigmas.

2. A model for the production of lepto-hadrons is obtained from an effective action describing the strong and electromagnetic interactions between lepto-hadrons. The simplest model is sigma model describing the interaction between lepto-nucleons, lepto-pion and a hypothetical scalar particle σ_L [29]. This model realizes lepto-pion field as a divergence of the axial current and gives the standard relation between $f(\pi_L), g$ and m_{ex}. All couplings of the model are related to the masses of e_{ex}, π_L and σ_L. The generation of lepto-pion vacuum expectation value in the proposed manner takes place via triangle anomaly diagrams in the external electromagnetic field.

3. If needed the model can be generalized to contain terms describing also other lepto-hadrons. The generalized model should contain also vector bosons ρ_L and ω_L as well as pseudoscalars η_L and η'_L and radial excitations of π_L and σ_L. An open question is whether also η and η' generate vacuum expectation value proportional to $E \cdot B$. Actually all these states appear as 3-fold degenerate for the minimal color representation content of the theory.

4. The following observations are useful for what follows.

   i) Ortopositronium decay width anomaly gives the estimate $f(\pi_L) \sim N_c \cdot 7.9\ keV$ and from this one can deduce an upper bound for lepto-pion production cross section in an external electromagnetic field. The calculation of lepto-pion production cross section shows that lepto-pion production cross section is somewhat smaller than the upper bound for the observed anomalous e^+e^- production cross section, even when one tunes the values of the various parameters. This is consistent with the idea that lepto-nucleon pairs, with lepto-nucleon mass being only slightly larger than electron mass, are in question.

   ii) Also the direct production of the lepto-nucleon pairs via the interaction term $gcos(\theta_e)\bar{e}_{ex}\gamma_5 e_{ex}\pi_L(cl)$ is possible but gives rise to continuum mass squared spectrum rather than resonant structures. The direct production of the pairs via the interaction term $gsin(\theta_e)\bar{e}\gamma_5 e_{ex}\pi_L(cl)$ from is much slower process than the production via the meson decays and does not give rise to resonant structures since Also the production via the $\bar{e}e_{ex}$ decay of virtual lepto-pion created from classical field is slow process since it involves $sin^2(\theta_e)$.

   iii) e^+e^- production can also proceed also via the creation of many particle states. The simplest candidates are $V_L + \pi_L$ states created via $\partial_\alpha \pi_L V^\alpha \pi_L(class)$ term in action and $\sigma_L + \pi_L$ states created via the the $k\sigma_L\pi_L\pi_L(class)$ term in the sigma model action. The production cross section via the decays of vector mesons is certainly very small since the production vertex involves the inner product of vector boson 3 momentum with its polarization vector and the situation is nonrelativistic.

iv) If the strong decay of σ_L to lepto-mesons is kinematically forbidden (this is not suggested by the experience with the ordinary hadron physics), the production rate for σ_L meson is large since the coupling k turns out to be given by $k = (m^2_{\sigma_L} - m^2_{\pi_L})/2f(\pi_L)$ and is anomalously large for the value of $f(\pi_L) \geq 7.9 \cdot N_c\ keV$ derived from ortopositronium anomaly: $k \sim 336m(\pi_L)/N_c$ for $f(\pi_L) \sim N_c \cdot 7.9\ keV$. The resulting additional factor in the production cross section compensates the reduction factor coming from two-particle phase space volume. Despite this the estimate for the production cross section of anomalous e^+e^- pairs is roughly by a factor $1/N_c^2$ smaller than the maximum experimental cross section. The radiative corrections are huge and should give the dominant contribution to the cross section. It is however questionable very the assumed small lepto-hadronic decay width and mass of σ_L is consistent with the extremely strong interactions of σ_L.

2.7.2 Simplest sigma model

A detailed description of the sigma model can be found in [29] and it suffices to outline only the crucial features here.

1. The action of lepto-hadronic sigma model reads as

$$\begin{aligned}
L &= L_S + c\sigma_L \ , \\
L_S &= \bar{\psi}_L(i\gamma^k\partial_k + g(\sigma_L + i\pi_L\cdot\tau\gamma_5))\psi_L + \frac{1}{2}((\partial\pi_L)^2 + (\partial\sigma_L)^2) \\
&- \frac{\mu^2}{2}(\sigma_L^2 + \pi_L^2) - \frac{\lambda}{4}(\sigma_L^2 + \pi_L^2)^2 \ .
\end{aligned} \tag{36}$$

   π_L is isospin triplet and σ_L isospin singlet. ψ_L is isospin doublet with electro-weak quantum numbers of electron and neutrino (e_{ex} and ν_{ex}). The model allows $so(4)$ symmetry. Vector current is conserved but for $c \neq 0$ axial current generates divergence, which is proportional to pion field: $\partial^\alpha A_\alpha = -c\pi_L$.

2. The presence of the linear term implies that σ_L field generates vacuum expectation value $\langle 0|\sigma_L|0\rangle = v$. When the action is written in terms of new quantum field $\sigma'_L = \sigma_L - v$ one has

$$\begin{aligned}
L &= \bar{\psi}_L(i\gamma^k\partial_k + m + g(\sigma'_L + i\pi_L\cdot\tau\gamma_5))\psi_L + \frac{1}{2}((\partial\pi_L)^2 + (\partial\sigma'_L)^2) \\
&- \frac{1}{2}m^2_{\sigma_L}(\sigma'_L)^2 - \frac{m^2_{\pi_L}}{2}\pi_L^2 \\
&- \lambda v\sigma'_L((\sigma'_L)^2 + \pi_L^2) - \frac{\lambda}{4}((\sigma'_L)^2 + \pi_L^2)^2 \ ,
\end{aligned} \tag{37}$$

   The masses are given by

$$\begin{aligned}
m^2_{\pi_L} &= \mu^2 + \lambda v^2 \ , \\
m^2_{\sigma_L} &= \mu^2 + 3\lambda v^2 \ , \\
m &= -gv \ .
\end{aligned} \tag{38}$$

   These formulas relate the parameters μ, v, g to lepto-hadrons masses.

3. The requirement that σ'_L has vanishing vacuum expectation implies in Born approximation

$$c - \mu^2 v - \lambda v^3 = 0 \ , \tag{39}$$

which implies

$$\begin{aligned} f(\pi_L) &= -v = -\frac{c}{m^2(\pi_L)} \ , \\ m_{ex} &= g f(\pi_L) \ . \end{aligned} \tag{40}$$

Note that e_{ex} and ν_{ex} are predicted to have identical masses in this approximation. The value of the strong coupling constant g of lepto-hadronic physics is indeed strong from $m_{ex} > m_e$ and $f(\pi_L) < N_c \cdot 10$ keV.

4. A new feature is the generation of the lepto-pion vacuum expectation value in an external electromagnetic field (of course, this is possible for the ordinary pion field, too!). The vacuum expectation is generated via the triangle anomaly diagram in a manner identical to the generation of a non-vanishing photon-photon decay amplitude and is proportional to the instanton density of the electromagnetic field. By redefining the pion field as a sum $\pi_L = \pi_L(cl) + \pi'_L$ one obtains effective action describing the creation of the lepto-hadrons in strong electromagnetic fields.

5. As far as the production of $\sigma_L \pi_L$ pairs is considered, the interaction term $\lambda v \sigma'_L \pi_L^2$ is especially interesting since it leads to the creation of $\sigma_L \pi_L$ pairs via the interaction term $k \lambda v \sigma'_L \pi_L(qu) \pi_L(cl)$.

   The coefficient of this term can be expressed in terms of the lepto-meson masses and $f(\pi_L)$:

$$\begin{aligned} k &\equiv 2\lambda v = \frac{m^2_{\sigma_L} - m^2_{\pi_L}}{2f(\pi_L)} = x m_{\pi_L} \ , \\ x &= \frac{1}{2}(\frac{m^2_{\sigma_L}}{m^2_{\pi_L}} - 1)\frac{m_{\pi_L}}{f(\pi_L)} \ . \end{aligned} \tag{41}$$

   The large value of the coupling deriving from $f(\pi_L) = N_c \cdot 7.9\ keV)$ compensates the reduction of the production rate coming from the smallness of two-particle phase space volume as compared with single particle-phase space volume but fails to produce large enough production cross section. The large value of $g(\sigma_L, \sigma_L, \sigma_L) = g(\sigma_L, \pi_L, \pi_L)$ however implies that the radiative contribution to the production cross section coming from the emission of a virtual sigma in the production vertex is much larger than the lowest order production cross section and with a rather small value of the relative $\sigma_L - \pi_L$ mass difference correct order of magnitude of cross section should be possible.

2.8 Classical model for lepto-pion production

The nice feature of both quantum and classical model is that the production amplitudes associated with all lepto-hadron production reactions in external electromagnetic field are proportional to the lepto-pion production amplitude and apart from phase space volume factors production cross sections are expected to be given by lepto-pion production cross section. Therefore it makes sense to construct a detailed model for lepto-pion production despite the fact that lepto-pion decays probably contribute only a very small fraction to the observed e^+e^- pairs.

2.8.1 General considerations

Angular momentum barrier makes the production of lepto-mesons with orbital angular momentum $L > 0$ improbable. Therefore the observed resonances are expected to be $L = 0$ pseudoscalar states. Lepto-pion production has two signatures which any realistic model should reproduce.

1. Data are consistent with the assumption that states are produced at rest in cm frame.

2. The production probability has a peak in a narrow region of velocities of colliding nucleus around the velocity needed to overcome Coulomb barrier in head on collision. The relative width of the velocity peak is of order $\Delta\beta/\beta \simeq \cdot 10^{-2}$ [36]. In Th-Th system [36] two peaks at projectile energies 5.70 MeV and 5.75 MeV per nucleon have been observed. This suggests that some kind of diffraction mechanism based on the finite size of nuclei is at work.
   In this section a model treating nuclei as point like charges and nucleus-nucleus collision purely classically is developed. This model yields qualitative predictions in agreement with the signature 1) but fails to reproduce the possible diffraction behavior although one can develop argument for understanding the behavior above Coulomb wall.

The general expression for the amplitude for creation of lepto-pion in external electric and magnetic fields has been derived in Appendix. Let us now specialize to the case of heavy ion collision. We consider the situation, where the scattering angle of the colliding nucleus is measured. Treating the collision completely classically we can assume that collision occurs with a well defined value of the impact parameter in a fixed scattering plane. The coordinates are chosen so that target nucleus is at rest at the origin of the coordinates and colliding nucleus moves in z-direction in y=0 plane with velocity β. The scattering angle of the scattered nucleus is denoted by α, the velocity of the lepto- pion by v and the direction angles of lepto-pion velocity by (θ, ϕ).

The minimum value of the impact parameter for the Coulomb collision of point like charges is given by the expression

$$\begin{aligned} b &= \frac{b_0 cot(\alpha/2)}{2} \ , \\ b_0 &= \frac{2Z_1Z_2\alpha_{em}}{M_R\beta^2} \ , \end{aligned} \quad (42)$$

where b_0 is the expression for the distance of the closest approach in head on collision. M_R denotes the reduced mass of the nucleus-nucleus system.

To estimate the amplitude for lepto-pion production the following simplifying assumptions are made.

1. Nuclei can be treated as point like charges. This assumption is well motivated, when the impact parameter of the collision is larger than the critical impact parameter given by the sum of radii of the colliding nuclei:

   $$b_{cr} = R_1 + R_2 \ . \quad (43)$$

   For scattering angles that are sufficiently large the values of the impact parameter do not satisfy the above condition in the region of the velocity peak. p-Adic considerations lead to the conclusion that nuclear condensation level corresponds to prime $p \sim 2^k$, $k = 113$ (k is prime). This suggest that nuclear radius should be replaced by the size $L(113)$ of the p-adic convergence cube associated with nucleus (see the chapter "TGD and Nuclear Physics": $L(113) \sim 1.7 \cdot 10^{-14}\ m$ implies that cutoff radius is $b_{cr} \sim 2L(113) \sim 3.4 \cdot 10^{-14}\ m$.

2. Since the velocities are non-relativistic (about $0.12c$) one can treat the motion of the nuclei non-relativistically and the non-retarded electromagnetic fields associated with the exactly known classical orbits can be used. The use of classical orbit doesn't take into account recoil effect caused by lepto-pion production. Since the mass ratio of lepto-pion and the reduced mass of heavy nucleus system is of order 10^{-5} the recoil effect is however negligible.

3. The model simplifies considerably, when the orbit is idealized with a straight line with impact parameter determined from the condition expressing scattering angle in terms of the impact parameter. This approximation is certainly well founded for large values of impact parameter. For small values of impact parameter the situation is quite different and an interesting problem is whether the contributions of long range radiation fields created by accelerating nuclei in head-on collision could give large contribution to lepto-pion production rate. On the line connecting the nuclei the electric part of the radiation field created by first nucleus is indeed parallel to the magnetic part of the radiation field created by second nucleus. In this approximation the instanton density in the rest frame of the target nucleus is just the scalar product of the Coulombic electric field E of the target nucleus and of the magnetic field B of the colliding nucleus obtained by boosting it from the Coulomb field of nucleus at rest.

2.8.2 Expression of the classical cross section

First some kinematical notations. Lepto-pion four-momentum in the rest system of target nucleus is given by the following expression

$$\begin{aligned} p &= (p_0, \bar{p}) = m\gamma_1(1, vsin(\theta)cos(\phi), vsin(\theta)sin(\phi), vcos(\theta)) \ , \\ \gamma_1 &= 1/(1-v^2)^{1/2} \ . \end{aligned} \quad (44)$$

The velocity and Lorentz boost factor of the projectile nucleus are denoted by β and $\gamma = 1/\sqrt{1-\beta^2}$.

The double differential cross section in the classical model can be written as

$$\begin{aligned} d\sigma &= dP2\pi bdb \ , \\ dP &= K|A(b,p)|^2 d^3n \ , per d^3n = V\frac{d^3p}{(2\pi)^3} \ , \\ K &= (Z_1Z_2)^2(\alpha_{em})^4 \times N_c^2(\frac{m(\pi_L}{f(\pi_L)})^2\frac{1}{2\pi^{13}} \ , \\ A(b,p) &= N_0\frac{4\pi}{Z_1Z_2\alpha_{em}} \times U(b,p) \ , \\ U(b,p) &= \int e^{ip\cdot x}E\cdot Bd^4x \ , \\ N_0 &= \frac{(2\pi)^7}{i} \ . \end{aligned} \quad (45)$$

where b denotes impact parameter. The formula generalizes the classical formula for the cross section of Coulomb scattering. In the calculation of the total cross section one must introduce some cutoff radii and the presence of the volume factor V brings in the cutoff volume explicitly (particle in the box description for lepto-pions). Obviously the cutoff length must be longer than lepto-pion Compton length. Normalization factor N_0 has been introduced in order to extract out large powers of 2π.

From this one obtains differential cross section as

$$
\begin{aligned}
d\sigma &= P2\pi b db \ , \\
P &= \int K|A(b,p)|^2 V\frac{d^3p}{(2\pi)^3} \ , \ .
\end{aligned} \qquad (46)
$$

The first objection is the need to explicitly introduce the reaction volume: this obviously breaks manifest Lorentz invariance. The cross section was estimated in the earlier version of the model [16] and turned to be too small by several orders of magnitude. This inspired the idea that constructive interference for the production amplitudes for different values of impact parameter could increase the cross section.

2.9 Quantum model for lepto-pion production

There are good reasons for considering the quantum model. First, the lepto-pion production cross section is by several orders of magnitude too small in classical model. Secondly, in Th-Th collisions there are indications about the presence of two velocity peaks with separation $\delta\beta/\beta \sim 10^{-2}$ [36] and this suggests that quantum mechanical diffraction effects might be in question. These effects could come from the upper and/or lower length scale cutoff and from the delocalization of the wave function of incoming nucleus.

The question is what quantum model means. The most natural thing is to start from Coulomb scattering and multiply Coulomb scattering amplitude for a given impact parameter value b with the amplitude for lepto-pion production. This because the classical differential cross section given by $2\pi bdb$ in Coulomb scattering equals to the quantum cross section. One might however argue that on basis of $S = 1 + T$ decomposition of S-matrix the lowest order contribution to lepto-pion production in quantum situation corresponds to the absence of any scattering. The lepto-pion production amplitude is indeed non-vanishing also for the free motion of nuclei. The resolution of what looks like a paradox could come from many-sheeted space-time concept: if no scattering occurs, the space-time sheets representing colliding nuclei do not touch and all and there is no interference of em fields so that there is no lepto-pion production. It turns however that lowest order contribution indeed corresponds to the absence of scattering in the model that works.

2.9.1 Two possible approaches

One can imagine two approaches to the construction of the model for production amplitude in quantum case.

The first approach is based on eikonal approximation [61]. Eikonal approximation applies at high energy limit when the scattering angle is small and one can approximate the orbit of the projectile with a straight orbit.

The expression for the scattering amplitude in eikonal approximation reads as

$$
\begin{aligned}
f(\theta,\phi) &= \frac{k}{2\pi i}\int d^2b exp(-ik\cdot b)exp(i\xi(b)) - 1) \ , \\
\xi(b) &= \frac{-m}{k\hbar^2}\int_{z=-\infty}^{z=\infty} dzV(z,b) \ , \\
\frac{d\sigma}{d\Omega} &= |f^2| \ .
\end{aligned} \qquad (47)
$$

as one expands the exponential in lowest in spherically symmetric potential order one obtains the

$$f(\theta,\phi) \simeq -\frac{m}{2\pi\hbar^2}\int J_0(k_T b)\xi(b) b db \quad . \tag{48}$$

The challenge is to find whether it is possible to generalize this expression so that it applies to the production of lepto-pions.

1. The simplest guess is that one should multiply the eikonal amplitude with the dimensionless amplitude $A(b)$:

$$\begin{aligned} f(\theta,\phi) &\rightarrow f(\theta,\phi,p) = \frac{k}{2\pi i}\int d^2 b exp(-ik\cdot b) exp(i\xi(b)) - 1) A(b,p) \\ &\simeq -\frac{m}{2\pi\hbar^2}\int J_0(k_T b)\xi(b) A(b,p) b db \quad . \end{aligned} \tag{49}$$

2. Amplitude squared must give differential cross section for lepto-pion production and scattering

$$\begin{aligned} d\sigma &= |f(\theta,\phi,p)|^2 d\Omega d^3 n \quad , \\ d^3 n &= V d^3 p \quad . \end{aligned} \tag{50}$$

   This requires an explicit introduction of a volume factor V via a spatial cutoff. This cutoff is necessary for the coordinate z in the case of Coulomb potential, and would have interpretation in terms of a finite spatio-temporal volume in which the space-time sheets of the colliding particles are in contact and fields interfere.

3. There are several objections against this approach. The loss of a manifest relativistic invariance in the density of states factor for lepto-pion does not look nice. One must keep count about the scattering of the projectile which means a considerable complication from the point of view of numerical calculations. In classical picture for orbits the scattering angle in principle is fixed once impact parameter is known so that the introduction of scattering angles does not look logical.

Second approach starts from the classical picture in which each impact parameter corresponds to a definite scattering angle so that the resulting amplitude describes lepto-pion production amplitude and says nothing about the scattering of the projectile. This approach is more in spirit with TGD since classical physics is exact part of quantum TGD and classical orbit is absolutely real from the point of view of lepto-pion production amplitude.

1. The counterpart of the eikonal exponent has interpretation as the exponent of classical action associated with the Coulomb interaction

$$S(b) = \int_\gamma V ds \tag{51}$$

   along the orbit γ of the particle, which can be taken also as a real classical orbit but will be approximated with rectilinear orbit in sequel.

2. The first guess for the production amplitude is

$$\begin{aligned} f(p) &= \int d^2bexp(-i\Delta k(b)\cdot b)exp[\frac{i}{\hbar}S(b)]A(b,p) \\ &= \int J_0(k_T(b)b)(1+\frac{i}{\hbar}\int_{z=-a}^{z=a} dzV(z,b)+..)A(b,p) \ . \end{aligned} \quad (52)$$

Δk is the change of the momentum in the classical scattering and in the scattering plane. The cutoff $|z|\leq a$ in the longitudinal direction corresponds to a finite imbedding space volume inside which the space-time sheets of target and projectile are in contact.

3. The production amplitude is non-trivial even if the interaction potential vanishes being given by

$$f(p) = \int d^2bexp(-ik\cdot b))A(b,p) = 2\pi int J_0(k_T(b)b)\times A(b,p)bdb \ . \quad (53)$$

This formula can be seen as a generalization of quantum formula in the sense that incoherent integral over production probabilities at various values of b is replaced by an integral over production amplitude over b so that interference effects become possible.

4. This result could be seen as a problem. On basis of $S=1+iT$ decomposition corresponding to free motion and genuine interaction, one could argue that since the exponent of action corresponds to S, $A(p,b)$ vanishes when the space-time sheets are not in contact. The improved guess for the amplitude is

$$\begin{aligned} f(p) &= \int d^2bexp(-ik\cdot b)exp(\frac{i}{\hbar}S(b))A(b,p) \\ &= \int J_0(k_T(b)b)(\frac{i}{\hbar}\int_{z=-a}^{z=a} V(z,b)+..)A(b,p) \ . \end{aligned} \quad (54)$$

This would mean that there would be no classical limit when coherence is assumed to be lost. At this stage one must keep mind open for both options.

5. The dimension of $f(p)$ is $L^2/\hbar$

$$d\sigma = |f(p)|^2\frac{d^3p}{2E_p(2\pi)^3} \ . \quad (55)$$

has correct dimension. This model will be considered in sequel. The earlier work in [16] was however based on the first option.

2.9.2 Production amplitude

The Fourier transform of $E\cdot B$ can be expressed as a convolution of Fourier transforms of E and B and the resulting expression for the amplitude reduces by residue calculus (see APPENDIX) to the following general form

$$
\begin{aligned}
A(b,p) &\equiv N_0 \times \frac{4\pi}{Z_1 Z_2 \alpha_{em}} \times U(b,p) = 2\pi i (CUT_1 + CUT_2) \ , \\
N_0 &= \frac{(2\pi)^7}{i} \ . \qquad (56)
\end{aligned}
$$

where nuclear charges are such that Coulomb potential is $1/r$. The motivation for the strange looking notation is to extract all powers of 2π so that the resulting amplitudes contain only factors of order unity.

The contribution of the first cut for $\phi \in [0, \pi/2]$ is given by the expression

$$
\begin{aligned}
CUT_1 &= D_1 \times \int_0^{\pi/2} exp(-\frac{b}{b_0} cos(\psi)) A_1 d\psi \ , \\
D_1 &= -\frac{1}{2}\frac{sin(\phi)}{sin(\theta)} \ , \ b_0 = \frac{\hbar}{m}\frac{\beta\gamma}{\gamma_1} \ , \\
A_1 &= \frac{A + iBcos(\psi)}{cos^2(\psi) + 2iCcos(\psi) + D} \ , \\
A &= sin(\theta)cos(\phi) \ , \ B = K \ , \\
C &= K\frac{cos(\phi)}{sin(\theta)} \ , \ D = -sin^2(\phi) - \frac{K^2}{sin^2(\theta)} \ , \\
K &= \beta\gamma(1 - \frac{v_{cm}}{\beta}cos(\theta)) \ , \ v_{cm} = \frac{2v}{1+v^2} \ .
\end{aligned}
\qquad (57)
$$

The definitions of the various kinematical variables are given in previous formulas. The notation is tailored to express the facts that A_1 is rational function of $cos(\psi)$ and that integrand depends exponentially on the impact parameter.

The expression for CUT_2 reads as

$$
\begin{aligned}
CUT_2 &= D_2 \times \int_0^{\pi/2} exp(i\frac{b}{b_1}cos(\psi)) A_2 d\psi \ , \\
D_2 &= -\frac{sin(\frac{\phi}{2})}{usin(\theta)} \times exp(-\frac{b}{b_2}) \ , \\
b_1 &= \frac{\hbar}{m}\frac{\beta}{\gamma_1} \ , \ b_2 = \frac{\hbar}{mb}\frac{1}{\gamma_1 \times sin(\theta)cos(\phi)} \\
A_2 &= \frac{Acos(\psi) + B}{cos^2(\psi) + Ccos(\psi) + D} \ , \\
A &= sin(\theta)cos(\phi)u \ , \ B = \frac{w}{v_{cm}} + \frac{v}{\beta}sin^2(\theta)[sin^2(\phi) - cos^2(\phi)] \ , \\
C &= 2i\frac{\beta w}{uv_{cm}}\frac{cos(\phi)}{sin(\theta)} \ , \ D = -\frac{1}{u^2}(\frac{sin^2(\phi)}{\gamma^2} + \beta^2(v^2 sin^2(\theta) - \frac{2vw}{v_{cm}})cos^2(\phi)) \\
&+ \frac{w^2}{v_{cm}^2 u^2 sin^2(\theta)} + 2i\frac{\beta v}{u}sin(\theta)cos(\phi) \ , \\
u &= 1 - \beta v cos(\theta) \ , \ w = 1 - \frac{v_{cm}}{\beta}cos(\theta) \ . \qquad (58)
\end{aligned}
$$

(59)

The denominator X_2 has no poles and the contribution of the second cut is therefore always finite. Again the expression is tailored to make clear the functional dependence of the integrand on $cos(\psi)$ and on impact parameter. Besides this the exponential damping makes in non-relativistic situation the integrand small everywhere expect in the vicinity of $cos(\Psi) = 0$ and for small values of the impact parameter.

Using the symmetries

$$\begin{aligned} U(b,p_x,-p_y) &= -U(b,p_x,p_y) \ , \\ U(b,-p_x,-p_y) &= \bar{U}(b,p_x,p_y) \ , \end{aligned} \tag{60}$$

of the amplitude one can calculate the amplitude for other values of ϕ.

CUT_1 gives the singular contribution to the amplitude. The reason is that the factor X_1 appearing in denominator of cut term vanishes, when the conditions

$$\begin{aligned} cos(\theta) &= \frac{\beta}{v_{cm}} \ , \\ sin(\phi) &= cos(\psi) \ , \end{aligned} \tag{61}$$

are satisfied. In forward direction this condition tells that z- component of the lepto-pion momentum in velocity center of mass coordinate system vanishes. In laboratory this condition means that the lepto-pion moves in certain cone defined by the value of its velocity. The condition is possible to satisfy only above the threshold $v_{cm} \geq \beta$.

For $K = 0$ the integral reduces to the form

$$CUT_1 = \frac{1}{2}cos(\phi)sin(\phi)\lim_{\varepsilon\to 0}\frac{\int_0^{\pi/2} exp(-\frac{cos(\psi)}{sin(\phi_0)})d\psi}{(sin^2(\phi) - cos^2\psi + i\varepsilon)} \ . \tag{62}$$

One can estimate the singular part of the integral by replacing the exponent term with its value at the pole. The integral contains two parts: the first part is principal value integral and second part can be regarded as integral over a small semicircle going around the pole of integrand in upper half plane. The remaining integrations can be performed using elementary calculus and one obtains for the singular cut contribution the approximate expression

$$\begin{aligned} CUT_1 &\simeq e^{-(b/a)(sin(\phi)/sin(\phi_0))}(\frac{ln(X)}{2} + \frac{i\pi}{2}) \ , \\ X &= \frac{((1+s)^{1/2} + (1-s)^{1/2})}{((1+s)^{1/2} - (1-s)^{1/2})} \ , \\ s &= sin(\phi) \ , \\ sin(\phi_0) &= \frac{\beta\gamma}{\gamma_1 m(\pi_L)a} \ . \end{aligned} \tag{63}$$

The principal value contribution to the amplitude diverges logarithmically for $\phi = 0$ and dominates over 'pole' contribution for small values of ϕ. For finite values of impact parameter the amplitude decreases exponentially as a function of ϕ.

If the singular term appearing in CUT_1 indeed gives the dominant contribution to the lepto-pion production one can make some conclusions concerning the properties of the production amplitude.

For given lepto-pion cm velocity v_{cm} the production associated with the singular peak is predicted to occur mainly in the cone $cos(\theta) = \beta/v_{cm}$: in forward direction this corresponds to the vanishing of the z-component of the lepto-pion momentum in velocity center of mass frame. Since the values of $sin(\theta)$ are of order .1 the transversal momentum is small and production occurs almost at rest in cm frame as observed. In addition, the singular production cross section is concentrated in the production plane ($\phi = 0$) due to the exponential dependence of the singular production amplitude on the angle ϕ and impact parameter and the presence of the logarithmic singularity. The observed lepto-pion velocities are in the range $\Delta v/v \simeq 0.2$ [36] and this corresponds to the angular width $\Delta\theta \simeq 34$ degrees.

2.9.3 Differential cross section in the quantum model

There are two options to consider depending on whether one uses $exp(iS)$ or $exp(iS) - 1$ to define the production amplitude.

1. For the $exp(iS)$ option the expression for the differential cross section reads in the lowest order as

$$\begin{aligned} d\sigma &= K|f_B|^2\frac{d^3p}{2E_p} \ , \\ f_B &\simeq i\int exp(-i\Delta k\cdot r)(CUT_1 + CUT_2)bdbdzd\phi \ , \\ K &= (Z_1Z_2)^2\alpha_{em}^4 N_c^2(\frac{m(\pi_L)}{f(\pi_L)})^2\frac{1}{(2\pi)^{15}} \ . \end{aligned} \tag{64}$$

   Here Δk is the momentum exchange in Coulomb scattering and a vector in the scattering plane so that the above described formula is obtained for the linear orbits.

2. For the $exp(iS) - 1$ option the differential production cross section for lepto-pion is in the lowest non-trivial approximation for the exponent of action S given by the expression

$$\begin{aligned} d\sigma &= K|f_B|^2\frac{d^3p}{2E_p} \ , \\ f_B &\simeq \int exp(-i\Delta k\cdot r)V(z,b)(CUT_1 + CUT_2)bdbdzd\phi \ , \\ V(z,b) &= \frac{1}{r} \ , \\ K &= (Z_1Z_2)^4\alpha_{em}^6 N_c^2(\frac{m(\pi_L)}{f(\pi_L)})^2\frac{1}{(2\pi)^{15}} \ . \end{aligned} \tag{65}$$

   Effectively the Coulomb potential is replaced with the product of the Coulomb potential and lepto-pion production amplitude $A(b,p)$. Since α_{em} is assumed to correspond to relate to its standard value by a scaling $\hbar_0/\hbar$ factor.

3. Coulomb potential brings in an additional $(Z_1Z_2\alpha_{em})^2$ factor to the differential cross section, which in the case of heavy ion scattering increases the contribution to the cross section by a factor of order 3×10^3 but reduces it by a factor of order 5×10^{-5} in the case of proton-antiproton scattering. The increase of $\hbar$ expected to be forced by the requirement that perturbation theory is not lost however reduces the contribution from higher orders in V. It should be possible to distinguish between the two options on basis of these differences.

The scattering amplitude can be reduced to a simpler form by using the defining integral representation

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} exp(-ixsin(\phi))d\phi$$

of Bessel functions.

1. For $exp(iS)$ option this gives

$$\begin{aligned} f_B &= 2\pi i \int J_0(\Delta k b)(CUT_1 + CUT_2) bdb \ , \\ \Delta k &= 2ksin(\frac{\alpha}{2}) \ , \ k = M_R\beta \ , \\ M_R &\simeq A_R m_p \ , \quad A_R = \frac{A_1A_2}{A_1+A_2} \ , \end{aligned} \tag{66}$$

   where the length scale cutoffs in various integrations are not written explicitly. The value of α can be deduced once the value of impact parameter is known in the case of the classical Coulomb scattering.

2. For $exp(iS) - 1$ option one has

$$\begin{aligned} f_B &= 2\pi i \int F(b) J_0(\Delta k b)(CUT_1 + CUT_2) bdb \ , \\ F(b \geq b_{cr}) &= 2\int dz \frac{1}{\sqrt{z^2+b^2}} = ln(\frac{\sqrt{a^2-b^2}+a}{b}) \ , \\ &. \end{aligned} \tag{67}$$

   Note that the factors K appearing in the different cross section are different in these to cases.

2.9.4 Calculation of the lepto-pion production amplitude in the quantum model

The details related to the calculation of the production amplitude can be found in appendix and it suffices to describe only the general treatment here. The production amplitude of the quantum model contains integrations over the impact parameter and angle parameter ψ associated with the cut. The integrands appearing in the definition of the contributions CUT_1 and CUT_2 to the scattering amplitude have simple exponential dependence on impact parameter. The function F appearing in the definition of the scattering amplitude is a rather slow varying function as compared to the Bessel function, which allows trigonometric approximation and for small values of scattering angle equals to its value at origin. This motivates the division of the impact parameter range into pieces so that F can approximated with its mean value inside each piece so that integration over cutoff parameters can be performed exactly inside each piece.

In Appendix the explicit expansion in power series with respect to impact parameter is derived by assuming $J_0(k_Tb) \simeq 1$ and $F(b) = F = constant$. These formulas can be easily generalized by assuming a piecewise constancy of these two functions. This means that the only the integration over the lepto-pion phase space must be carried out numerically.

CUT_1 becomes also singular at $cos(\theta) = \beta/v_{cm}$, $cos(\psi) = \sin(\phi)$. The singular contribution of the production amplitude can be extracted by putting $cos(\psi) = sin(\phi)$ in the arguments of the exponent functions appearing in the amplitude so that one obtains a rational function of $cos(\psi)$ and $sin(\psi)$ integrable analytically. The remaining nonsingular contribution can be integrated numerically.

2.9.5 Formula for the production cross section

In the case of heavy ion collisions the rectilinear motion is not an excellent approximation since the anomalous events are observed near Coulomb wall and $\beta \simeq .1$ holds true. Despite this this can be taken as a first approximation.

The expression for the differential cross section for lepto-pion production in heavy ion collisions is given by

$$d\sigma = KF^2|\int (CUT_1 + CUT_2)bdb|^2 \frac{d^3p}{2E} \ , \tag{68}$$

This expression and also the expressions of the integrals of CUT_1 and CUT_2 are calculated explicitly as powers series of the impact parameter in the Appendix.

1. For $exp(iS)$ option one has

$$\begin{aligned} K &= (Z_1Z_2)^2\alpha_{em}^4 N_c^2[\frac{m(\pi_L)}{f(\pi_L)}]^2\frac{1}{(2\pi)^{13}} \ , \\ F &= 1 \ . \end{aligned} \tag{69}$$

2. For $exp(iS) - 1$ option one has

$$\begin{aligned} K &= (Z_1Z_2)^4\alpha_{em}^6 N_c^2[\frac{m(\pi_L)}{f(\pi_L)}]^2\frac{1}{(2\pi)^{13}} \ , \\ F &= 2\langle\langle ln(\frac{\sqrt{a^2-b^2}+a}{b})\rangle\rangle \ . \end{aligned} \tag{70}$$

In the approximation that F is constant the two lowest order predictions are related by a scaling factor

$$R = (Z_1Z_2\alpha_{em})^2F^2 \ . \tag{71}$$

It is interesting to get a rough order of magnitude feeling about the situation assuming that the contributions of CUT_1 and CUT_2 are of order unity. For $Z_1 = Z_2 = 92$ and $m(\pi_L)/f(\pi_L) \simeq 1.5$ -as in the case of ordinary pion- one obtains following results. It must be emphasized that these estimates are extremely sensitive to the over all scaling of f_B and to the choice of the cutoff parameter a and cannot be taken too seriously.

1. From $\beta \simeq .1$ one has $b_0 \simeq .1/m(\pi_L)$. One can argue that the impact parameter cutoff $a = xb_0$ should satisfy $a \geq 1/m_{\pi_L}$ so that $x \geq 10$ should hold true.

2. For $expi(S) - 1$ option one has $K = 4.7 \times 10^{-6}$. From the classical model the allowed phase space volume is of order $\frac{1}{3}\Delta v^3 \sim 10^{-4}$. By using $a = m(\pi_L)$ as a cutoff and $m(\pi_L) \simeq 2m_e$ one obtains $\sigma \sim 4$ μb, which is of same order of magnitude as the experimental estimate 5 μb.

3. For $exp(iS)$ option one has $K = 1.2 \times 10^{-9}$ and the estimate for cross section is 1.1 nb for $a = 1/m(\pi_L)$. A correct order of magnitude is obtained by assuming $a = 5.5/m(\pi_L)$ and that a^4 scaling holds true. At larger values of impact parameter a^2 scaling sets on and would require $a \sim 30/m(\pi_L)$ which would correspond to .36 A and to atomic length scale. It is not possible to distinguish between the two options.

4. The singular contribution near to production plane at the cone $v_{cm}cos(\theta) = \beta$ is expected to enhance the total cross section. The strong sensitivity of the cross section to the choice of the cutoff parameter allows to reproduce the experimental findings easily and it would be important to establish strong bounds on the value of the impact parameter.

2.9.6 Dominating contribution to production cross section and diffractive effects

Consider now the behavior of the dominating singular contribution to the production amplitude at the cone $cos(\theta) = \beta/v_{cm}$ depending on b via the exponent factor . This amplitude factorizes into a product

$$
\begin{aligned}
f_{B,sing} &= K_0 a^2 B(\Delta k) A_{sing}(b,p) \ , \\
B(\Delta k) &= \int F(ax) J_0(\Delta k a x) exp(-\frac{sin(\phi)}{sin(\phi_0)} x) x dx \ , \\
&\sim \sqrt{\frac{2}{\pi \Delta k a}} \int F(ax) cos(\Delta k a x - \frac{\pi}{4}) exp(-\frac{sin(\phi)}{sin(\phi_0)} x) \sqrt{x} dx \ , \\
x &= \frac{b}{a} \ . \qquad (72)
\end{aligned}
$$

The factor $A_{sing}(b,p) \equiv (4\pi/(Z_1 Z_2 \alpha_{em}) U_{sing}(b,p)$ is the analytically calculable singular and dominating part of the lepto-pion production amplitude (see appendix) with the exponential factor excluded. The factor B is responsible for diffractive effects. The contribution of the peak to the total production cross section is of same order of magnitude as the classical production cross section.

At the peak $\phi \sim 0$ the contribution the exponent of the production amplitude is constant at this limit one obtains product of the Fourier transform of Coulomb potential with cutoffs with the production amplitude. One can calculate the Fourier transform of the Coulomb potential analytically to obtain

$$
\begin{aligned}
f_{B,sing} &\simeq 4\pi K_0 \frac{(cos(\Delta k a) - cos(\Delta k b_{cr}))}{\Delta k^2} CUT_1 \\
\Delta k &= 2\beta sin(\frac{\alpha}{2}) \ . \qquad (73)
\end{aligned}
$$

One obtains oscillatory behavior as a function of the collision velocity in fixed angle scattering and the period of oscillation depends on scattering angle and varies in wide limits.

The relationship between scattering angle α and impact parameter in Coulomb scattering translates the impact parameter cutoffs to the scattering angle cutoffs

$$
\begin{aligned}
a &= \frac{Z_1 Z_2 \alpha_{em}}{M_R \beta^2} cot(\alpha(min)/2) \ , \\
b_{cr} &= \frac{Z_1 Z_2 \alpha_{em}}{M_R \beta^2} cot(\alpha(max)/2) \ . \qquad (74)
\end{aligned}
$$

This gives for the argument Δkb of the Bessel function at lower and upper cutoffs the approximate expressions

$$\begin{aligned} \Delta ka &\simeq \frac{2Z_1Z_2\alpha_{em}}{\beta} \sim \frac{124}{\beta} \ , \\ \Delta kb_{cr} &\simeq x_0\frac{2Z_1Z_2\alpha_{em}}{\beta} \sim \frac{124x_0}{\beta} \ . \end{aligned} \tag{75}$$

The numerical values are for $Z_1 = Z_2 = 92$ (U-U collision). What is remarkable that the argument Δka at upper momentum cutoff does not depend at all on the value of the cutoff length. The resulting oscillation at minimum scattering angle is more rapid than allowed by the width of the observed peak: $\Delta\beta/\beta \sim 3\cdot 10^{-3}$ instead of $\Delta\beta/\beta \sim 10^{-2}$: of course, the measured value need not correspond to minimum scattering angle. The oscillation associated with the lower cutoff comes from $cos(2M_Rb_{cr}\beta sin(\alpha/2))$ and is slow for small scattering angles $\alpha < 1/A_R \sim 10^{-2}$. For $\alpha(max)$ the oscillation is rapid: $\delta\beta/\beta \sim 10^{-3}$.

In the total production cross section integrated over all scattering angles (or finite angular range) diffractive effects disappear. This might explain why the peak has not been observed in some experiments [36].

2.9.7 Cutoff length scales

Consider next the constraints on the upper cutoff length scale.

1. The production amplitude turns out to decrease exponentially as a function of impact parameter b unless lepto-pion is produced in scattering plane. The contribution of lepto-pions produced in scattering plane however gives divergent contribution to the total cross section integrated over all impact parameter values and upper cutoff length scale a is necessary. If one considers scattering with scattering angle between specified limits this is of course not a problem of classical model.

2. Upper cutoff length scale must be longer than the Compton length of lepto-pion.

3. Upper cutoff length scale a should be certainly smaller than the interatomic distance. For partially ionized atoms a more stringent upper bound for a is the size r of atom defined as the distance above which atom looks essentially neutral: a rough extrapolation from hydrogen atom gives $r \sim a_0/Z^{1/3} \sim 1.5\cdot 10^{-11}\ m$ (a_0 is Bohr radius of hydrogen atom). Therefore cutoff scale would be between Bohr radius $a_0/Z \sim .5\cdot 10^{-12}\ m$ and r. In the recent case however atoms are completely ionized so that cutoff length scale can be longer. It turns out that 10 A reproduces the empirical estimate for the cross section correctly.

2.9.8 Numerical estimate for the electro-pion production cross section

The numerical estimate for the electro-pion production cross section is carried out for thorium with $(Z = 90, A = 232)$. The value of the collision velocity of the incoming nucleus in the rest frame of the second nucleus is taken as $\beta = .1$. From the width $\delta v/v = .2$ of velocity distribution in the same frame the upper bound $\gamma \leq 1+\delta$, $\delta \simeq 2\times 10^{-3}$ for the Lorentz boost factor of electro-pion in cm system is deduced. The cutoff is necessary because energy conservation is not coded to the structure of the model.

As expected, the singular contribution from the cone $v_{cm}cos(\theta) = \beta$, $v_{cm} = 2v/(1+v^2)$ gives the dominating contribution to the cross section. This contribution is proportional to the value of

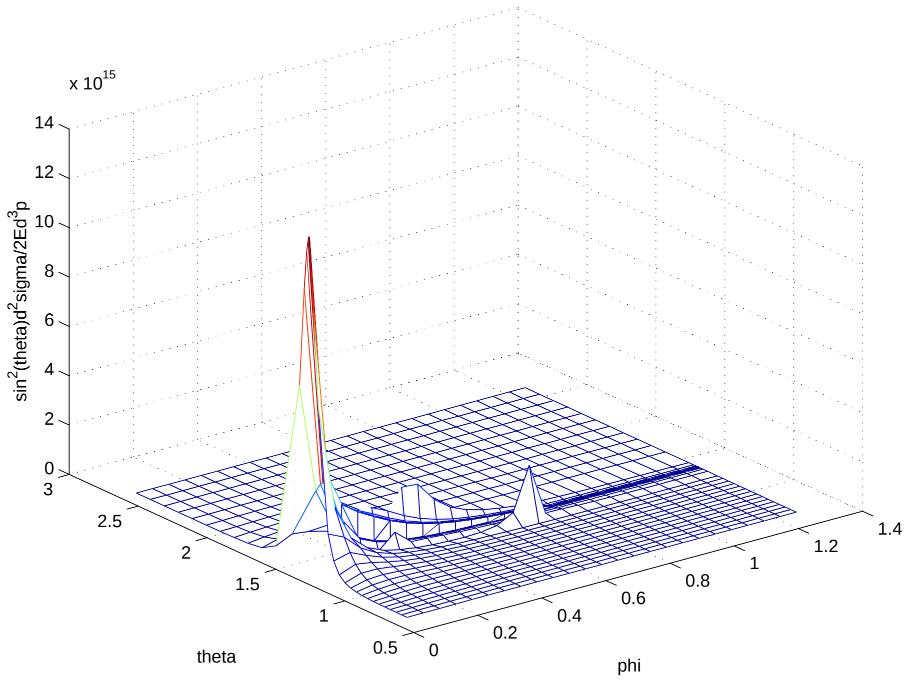

Figure 1: Differential cross section $sin^2(\theta) \times \frac{d^2\sigma}{2Ed^3p}$ for τ-pion production for $\gamma_1 = 1.0319 \times 10^3$ in the rest system of antiproton for $\delta = 1.5$. $m(\pi_\tau)$ defines the unit of energy and nb is the unit for cross section. The ranges of θ and ϕ are $(0, \pi)$ and $(0, \pi/2)$.

b_{max}^2 at the limit $\phi = 0$. Cutoff radius is taken to be $b_{max} = 150 \times \gamma_{cm}\hbar/m(\pi_e) = 1.04$ A. The numerical estimate for the cross section using the parameter values listed comes out as $\sigma = 5.6\ \mu$b to be compared with the rough experimental estimate of about 5 μb. The interpretation would be that the space-time sheet associated with colliding nuclei during the collision has this transversal size in cm system. At this space-time sheet the electric and magnetic fields of the nuclei interfere.

From this one can cautiously conclude that lepto-pion model is consistent with both electro-pion production and τ-pion production in proton antiproton collisions. One can of course criticize the large value of impact parameter and a good justification for 1 Angstrom should be found. One could also worry about the singular character of the amplitude making the integration of total cross section somewhat risky business using the rather meager numerical facilities available. The rigorous method to calculate the contribution near the singularity relies on stepwise halving of the increment $\Delta\theta$ as one approaches the singularity. The calculation gives essentially the same result as that with constant value of $\Delta\theta$. Hence it seems that one can trust on the result of calculation.

Figure 2. gives the differential production cross section for $\gamma_1 = 1.0319$. Obviously the differential cross section is strongly concentrated at the cone due to singularity of the production amplitude for fixed b.

The important conclusion is that the same model can reproduce the value of production cross section for both electro-pions explaining the old electron-positron anomaly of heavy ion collisions and τ-pions explaining the CDF anomaly of proton-antiproton collisions at cm energy $\sqrt{s} = 1.96$ TeV (to be discussed later) with essentially same and rather reasonable assumptions (do not however forget the large maximal value of the impact parameter!).

In the case of electro-pions one must notice that depending on situation the final states are gamma pairs for the electron-pion with mass very nearly equal to electron mass. In the case of neutral tau-pion the strong decay to three p-adically scaled down versions of τ-pion proceeds faster or at least rate comparable to that for the decay to gamma pair. For higher mass variants of electro-pion for which there is evidence (for instance, one with mass 1.6 MeV) the final states are dominated by electron-positron pairs. This is true if the primary decay products are electro-baryons of form (say) $e_{ex} = e_8\nu_8\nu_{c,8}$ resulting via electro-strong decays instead of electrons and having slightly larger mass than electron. Otherwise the decay to gamma pair would dominate also the decays of higher mass states. A small magnetic moment type coupling between e, e_{ex} and electro-gluon field made possible by the color octet character of colored leptons induces the mixing of e and e_{ex} so that e_{ex} can transform to e by the emission of photon. The anomalous magnetic moment of electron poses restrictions on the color magnetic coupling.

2.9.9 $e^+_{ex}e^-_{ex}$ pairs from lepto-pions or e^+e^- pairs from lepto-sigmas?

If one assumes that anomalous e^+e^- pairs correspond to lepto-nucleon pairs, then lepto-pion production cross section gives a direct estimate for the production rate of e^+e^- pairs. The results of the table 3 show that in case of 1.8 MeV state, the predicted cross section is roughly by a factor 5 smaller than the experimental upper bound for the cross section. Since this lepto-pion state is rather massive, positron decay width allows smaller $f(\pi_L)$ in this case and the production cross section could be larger than the estimate used by the $1/f(\pi_L)^2$ proportionality of the cross section. Both the simplicity and predictive power of this option and the satisfactory agreement with the experimental data suggest that this option provides the most plausible explanation of the anomalous e^+e^- pairs.

N	$Op/10^{-3}$	$\Gamma(\pi_L)/keV$	$\sigma(\pi_L)/\mu b$	$\sigma(\pi_L)/\mu b$
			$a = .01$	$a = .1$
1	1	.51	.13	1.4
3	1	.13	.04	.41
3	5	.73	.19	2.1

Table 2. The table summarizes lepto-pion lifetime and the upper bounds for lepto-pion (and lepto-nucleon pair) production cross sections for the lightest lepto-pion. N refers to the number of lepto-pion states and $Op = \Delta\Gamma/\Gamma$ refers to ortopositronium decay anomaly. The values of upper cutoff length a are in units of 10^{-10} m.

If one assumes that anomalous e^+e^- pairs result from the decays of lepto-sigmas, the value of e^+e^- production cross section can be estimated as follows. e^+e^- pairs are produced from via the creation of $\sigma_L\pi_L$ pairs from vacuum and subsequent decay σ_L to e^+e^- pairs. The estimate for (or rather for the upper bound of) $\pi_L\sigma_L$ production cross section is obtained as

$$\begin{aligned}
\sigma(e^+e^-) &\simeq X\sigma(\pi_L) \ , \\
X &= \frac{V_2}{V_1}(\frac{km_{\sigma_L}}{m^2_{\pi_L}})^2 \ , \\
\frac{V_2}{V_1} &= V_{rel} = \frac{v_{12}^3}{3(2\pi)^2} \sim 1.1 \cdot 10^{-5} \ , \\
\frac{k}{m_{pi_L}} &= \frac{(m_\sigma^2 - m^2_{\pi_L})}{2m_{\pi_L}f(\pi_L)} \ .
\end{aligned} \tag{76}$$

Here V_2/V_1 of two-particle and single particle phase space volumes. V_2 is in good approximation the product $V_1(cm)V_1(rel)$ of single particle phase space volumes associated with cm coordinate and relative coordinate and one has $V_2/V_1 \sim V_{rel} = \frac{v_{12}^3}{3(2\pi)^2)} \simeq 1.1 \cdot 10^{-5}$ if the maximum value of the relative velocity is $v_{12} \sim .1$.

Situation is partially saved by the anomalously large value of $\sigma_L \pi_L \pi_L$ coupling constant k appearing in the production vertex $k\sigma_L \pi_L \pi_L(class)$. Production cross section is very sensitive to the value of $f(\pi_L)$ and Op anomaly $\Delta\Gamma/\Gamma = 5 \cdot 10^{-3}$ gives upper bound 2 $\mu b/N_c^2$ for $a = 10^{-11}$ m, which is considerably smaller than the experimental upper bound 5 μb. The huge value of the $g(\pi_L, \pi_L, \sigma_L)$ and $g(\sigma_L, \sigma_L, \sigma_L)$, however implies that radiative corrections to the cross section given by σ exchange are much larger than the lowest order contribution to the cross section! If this is the case then lepto-sigma option might survive but perturbative approach probably would not make sense. On the other hand, one could argue that sigma model action should be regarded as an effective action giving only tree diagrams so that radiative corrections cannot save the situation. There are also purely physical counter arguments against lepto-sigma option: hadronic physics experience suggests that the mass of lepto-sigma is much larger than lepto-pion mass so that lepto-sigma becomes very wide resonance decaying strongly and having negligibly small branching ratio to e^+e^- pairs.

It must be emphasized that the estimates are very rough (the replacement of the integral over the angle α with rough upper bound, estimate for the phase space volume, the values of cutoff radii, the neglect of the velocity dependence of the production cross section, the estimate for the minimum scattering angle, ...). Also the measured production cross section is subject to considerable uncertainties (even the issue whether or not anomalous pairs are produced is not yet completely settled!).

2.9.10 Summary

The usefulness of the modeling lepto-pion production is that the knowledge of lepto-pion production rate makes it possible to estimate also the production rates for other lepto-hadrons and even for many particle states consisting of lepto-hadrons using some effective action describing the strong interactions between lepto-hadrons. One can consider two basic models for lepto-pion production. The models contain no free parameters unless one regards cutoff length scales as such. Classical model predicts the singular production characteristics of lepto-pion. Quantum model predicts several velocity peaks at fixed scattering angle and the distance between the peaks of the production cross section depends sensitively on the value of the scattering angle. Production cross section depends sensitively on the value of the scattering angle for a fixed collision velocity. In both models the reduction of the lepto-pion production rate above Coulomb wall could be understood as a threshold effect: for the collisions with impact parameter smaller than two times nuclear radius, the production amplitude becomes very small since $E \cdot B$ is more or less random for these collisions in the interaction region. The effect is visible for fixed sufficiently large scattering angle only. The value of the anomalous e^+e^- production cross section is of nearly the observed order of magnitude provided that e^+e^- pairs are actually lepto-nucleon pairs originating from the decays of the lepto-pions. Alternative mechanism, in which anomalous pairs originate from the creation of $\sigma_L \pi_L$ pairs from vacuum followed by the decay $\sigma_L \rightarrow e^+e^-$ gives too small production cross section by a factor of order $1/N_c^2$ in lowest order calculation. This alternative works only provided that radiative corrections give the dominant contribution to the production rate of $\pi_L \sigma_L$ pairs as is the case if $\pi_L \sigma_L$ mass difference is of order ten per cent. The existence of at least three colored leptons and family replication provide the most plausible explanation the appearance of several peaks.

The proposed models are certainly over idealizations: in particular the approximation that nuclear motion is free motion fails for those values of the impact parameter, which are most important in the classical model. To improve the models one should calculate the Fourier transform of $E \cdot B$ using the fields of nuclei for classical orbits in Coulomb field rather than free motion. The second improvement is related to the more precise modelling of the situation at length scales below b_{cr}, where nuclei do not behave like point like charges. A peculiar feature of the model from the point of view of standard physics is the appearance of the classical electromagnetic fields associated with the classical orbits of the colliding nuclei in the definition of the quantum model. This is in spirit with Quantum TGD: Quantum TGD associates a unique space-time surface (classical history) to a given 3-surface (counterpart of quantum state).

3 Further developments

This section represents further developments of leptohadron model which have emerged during years after the first version of the model published in International Journal of Theoretical Physics.

3.1 How to observe leptonic color?

The most obvious argument against lepto-hadrons is that their production via the decay of virtual photons to lepto-mesons has not been observed in hadronic collisions. The argument is wrong. Anomalously large production of low energy e^+e^- pairs [21, 22, 23, 24] in hadronic collisions has been actually observed. The most natural source for photons and e^+e^- pairs are lepto-hadrons. There are two possibilities for the basic production mechanism.

1. Colored leptons result directly from the decay of hadronic gluons. Internal consistency excludes this alternative.

2. Colored leptons result from the decay of virtual photons. This hypothesis is in accordance with the general idea that the QCD:s associated with different condensate levels of p-adic topological condensate do not communicate. More precisely, in TGD framework leptons and quarks correspond to different chiralities of configuration space spinors: this implies that baryon and lepton numbers are conserved exactly and therefore the stability of proton. In particular, leptons and quarks correspond to different Kac Moody representations: important difference as compared with typical unified theory, where leptons and quarks share common multiplets of the unifying group. The special feature of TGD is that there are several gluons since it is possible to associate to each Kac-Moody representation gluons, which are "irreducible" in the sense that they couple only to a single Kac Moody representation. It is clear that if the physical gluons are "irreducible" the world separates into different Kac Moody representations having their own color interactions and communicating only via electro-weak and gravitational interactions. In particular, no strong interactions between leptons and hadrons occur. Since colored lepton corresponds to colored ground state of Kac-Moody representations the gluonic color coupling between ordinary lepton and colored lepton vanishes.

If this picture is correct then lepto-hadrons are produced only via the ordinary electro-weak interactions: at higher energies via the decay of virtual photon to colored lepton pair and at low energies via the emission of lepto-pion by photon. Consider next various manners to observe the effects of lepton color.

1. Resonance structure in the photon-photon scattering and energy near lepto-pion mass is a unique signature of lepto-pion.

2. The production of lepto-mesons in strong classical electromagetic fields (of nuclei, for example) is one possibility. There are several important constraints for the production of lepto-pions in this kind of situation.

   i) The scalar product $E \cdot B$ must be large. Faraway from the source region this scalar product tends to vanish: consider only Coulomb field.

ii) The region, where $E \cdot B$ has considerable size cannot be too small as compared with lepto-pion de Broglie wavelength (large when compared with the size of nuclei for example). If this condition doesn't hold true the plane wave appearing in Fourier amplitude is essentially constant spatially and since the fields are approximately static the Fourier component of $E \cdot B$ is expressible as a spatial divergence, which reduces to a surface integral over a surface faraway from the source region. Resulting amplitude is small since fields in faraway region have essentially vanishing $E \cdot B$.

iii) If fields are exactly static, then energy conservation prohibits lepto-hadron production.

3. Also the production of $e^+_{ex}e^-_{ex}$ and $e^+e^-_{ex}$ pairs in nuclear electromagnetic fields with non-vanishing $E \cdot B$ is possible either directly or as decay products of lepto-pions. In thedirect production, the predicted cross section is small due to the presence of two-particle phase space factor. One signature of e^-_{ex} is emission line accompanying the decay $e^-_{ex} \rightarrow e^- + \gamma$. The collisions of nuclei in highly ionized (perhaps astrophysical) plasmas provide a possible source of leptobaryons.

4. The interaction of quantized em field with classical electromagnetic fields is one experimental arrangement to come into mind. The simplest arrangement consisting of linearly polarized photons with energy near lepto-pion mass plus constant classical em field does not however work. The direct production of $\pi_L - \gamma$ pairs in rapidly varying classical electromagnetic field with frequency near lepto-pion mass is perhaps a more realistic possibility . An interesting possibility is that violent collisions inside astrophysical objects could lead to gamma ray bursts via the production of pions and lepto-pions in rapidly varying classical E and B fields.

5. In the collisions of hadrons, virtual photon produced in collision can decay to lepto-hadrons, which in turn produce lepto-pions decaying to lepto-nucleon pairs. As already noticed, anomalous production of low energy e^+e^- pairs (actually lepto-nucleon pairs!) [21] in hadronic collisions has been observed.

6. $e-\nu_e$ and $e-\bar{\nu}_e$ scattering at energies below one MeV provide a unique signature of lepto-pion. In $e - \bar{\nu}_e$ scattering π_L appears as resonance.

7. If leptonic color coupling strength has sufficiently small value in the energy range at which lepto-hadronic QCD exists, e^+e^- annihilation at energies above few MeV should produce colored pairs and lepto-hadronic counterparts of the hadron jets should be observed. The fact that nothing like this has been observed, suggests that lepto-hadronic coupling constant evolution does not allow the perturbative QCD phase.

3.2 New experimental evidence

After writing this chapter astrophysical support for the notion of lepto-pions has appeared. There is also experimental evidence for the existence of colored muons

3.2.1 Could lepto-hadrons correspond to dark matter?

The proposed identification of cosmic strings (in TGD sense) as the ultimate source of both visible and dark matter discussed in [D4] does not exclude the possibility that a considerable portion of topologically condensed cosmic strings have decayed to some light particles. In particular, this could be the situation in the galactic nuclei.

The idea that lepto-hadrons might have something to do with the dark matter has popped up now and then during the last decade but for some reason I have not taken it seriously. Situation changed towards the end of the year 2003. There exist now detailed maps of the dark matter in the center of galaxy and it has been found that the density of dark matter correlates strongly with the intensity of monochromatic photons with energy equal to the rest mass of electron [38].

The only explanation for the radiation is that some yet unidentified particle of mass very nearly equal to $2m_e$ decays to an electron positron pair. Electron and positron are almost at rest and this implies a high rate for the annihilation to a pair of gamma rays. A natural identification for the particle in question would be as a lepto-pion (or rather, electro-pion). By their low mass lepto-pions, just like ordinary pions, would be produced in high abundance, in lepto-hadronic strong reactions and therefore the intensity of the monochromatic photons resulting in their decays would serve as a measure for the density of the lepto-hadronic matter. Also the presence of lepto-pionic condensates can be considered.

These findings force to take seriously the identification of the dark matter as lepto-hadrons. This is however not the only possibility. The TGD based model for tetra-neutrons discussed in [F8] is based on the hypothesis that mesons made of scaled down versions of quarks corresponding to Mersenne prime M_{127} (ordinary quarks correspond to $k = 107$) and having masses around one MeV could correspond to the color electric flux tubes binding the neutrons to form a tetra-neutron. The same force would be also relevant for the understanding of alpha particles.

There are also good theoretical arguments for why lepto-hadrons should be dark matter in the sense of having a non-standard value of Planck constant.

1. Since particles with different Planck constant correspond to different pages of the book like structure defining the generalization of the imbedding space, the decays of intermediate gauge bosons to colored excitations of leptons would not occur and would thus not contribute to their decay widths.

2. In the case of electro-pions the large value of the coupling parameter $Z_1Z_2\alpha_{em} > 1$ combined with the hypothesis that a phase transition increasing Planck constant occurs as perturbative QFT like description fails would predict that electro-pions represent dark matter. Indeed, the power series expansion of the $exp(iS)$ term might well fail to converge in this case since S is proportional to Z_1Z_2. For τ-pion production one has $Z_1 = -Z_2 = 1$ and in this case one can consider also the possibility that τ-pions are not dark in the sense of having large Planck constant. Contrary to the original expectations darkness does not affect the lowest order prediction for the production cross section of lepto-pion.

The proposed identification raises several questions.

1. Why the ratio of the lepto-hadronic mass density to the mass density of the ordinary hadrons would be so high, of order 7? Could an entire hierarchy of asymptotically non-free QCDs be responsible for the dark matter so that lepto-hadrons would explain only a small portion of the dark matter?

2. Under what conditions one can regard lepto-hadronic matter as a dark matter? Could short life-times of lepto-hadrons make them effectively dark matter in the sense that there would be no stable enough atom like structures consisting of say charged lepto-baryons bound electromagnetically to the ordinary nuclei or electrons? But what would be the mechanism producing lepto-hadrons in this case (nuclear collisions produce lepto-pions only under very special conditions)?

3. What would be the role of the many-sheeted space-time: could lepto-hadrons and atomic nuclei reside at different space-time sheets so that lepto-baryons could be long-lived? Could dark matter quite generally correspond to the matter at different space-time sheets and thus serve as a direct signature of the many-sheeted space-time topology?

3.2.2 Experimental evidence for colored muons

Also μ and τ should possess colored excitations. About fifteen years after this prediction was made. Direct experimental evidence for these states finally emerges (the year I am adding this comment is 2007) [53, 54]. The mass of the new particle, which is either scalar or pseudoscalar, is 214.4 MeV whereas muon mass is 105.6 MeV. The mass is about 1.5 per cent higher than two times muon mass. The proposed interpretation is as a light Higgs. I do not immediately resonate with this interpretation although p-adically scaled up variants of also Higgs bosons live happily in the fractal Universe of TGD. The most natural TGD inspired interpretation is as a pion like bound state of colored excitations of muon completely analogous to lepto-pion (or rather, electro-pion).

Scaled up variants of QCD appear also in nuclear string model [F8, 17], where scaled variant of QCD for exotic quarks in p-adic length scale of electron is responsible for the binding of ^4He nuclei to nuclear strings. One cannot exclude the possibility that the fermion and anti-fermion at the ends of color flux tubes connecting nucleons are actually colored leptons although the working hypothesis is that they are exotic quark and anti-quark. One can of course also turn around the argument: could it be that lepto-pions are "lepto-nuclei", that is bound states of ordinary leptons bound by color flux tubes for a QCD in length scale considerably shorter than the p-adic length scale of lepton.

3.3 Evidence for τ-hadrons

The evidence for τ-leptons came in somewhat funny but very pleasant manner. During my friday morning blog walk, the day next to my birthday October 30, I found that Peter Woit had told in his blog about a possible discovery of a new long-lived particle by CDF experiment [55] emphasizing how revolutionary finding is if it is real. There is a detailed paper [59] with title *Study of multi-muon events produced in p-pbar collisions at* $\sqrt(s) = 1.96\ TeV$ by CDF collaboration added to the ArXiv October 29 - the eve of my birthday. I got even second gift posted to arXiv the very same day and reporting an anomalously high abundance of positrons in cosmic ray radiation [48]. Both of these article give support for basic predictions of TGD differentiating between TGD and standard model and its generalizations.

3.3.1 The first gift

A brief summary of Peter Woit about the finding gives good idea about what is involved.

The article originates in studies designed to determine the b-bbar cross-section by looking for events, where a b-bbar pair is produced, each component of the pair decaying into a muon. The b-quark lifetime is of order a picosecond, so b-quarks travel a millimeter or so before decaying. The tracks from these decays can be reconstructed using the inner silicon detectors surrounding the beam-pipe, which has a radius of 1.5 cm. They can be characterized by their impact parameter, the closest distance between the extrapolated track and the primary interaction vertex, in the plane transverse to the beam.

If one looks at events where the b-quark vertices are directly reconstructed, fitting a secondary vertex, the cross-section for b-bbar production comes out about as expected. On the other hand, if one just tries to identify b-quarks by their semi-leptonic decays, one gets a value for the b-bbar cross-section that is too large by a factor of two. In the second case, presumably there is some background being misidentified as b-bbar production.

The new result is based on a study of this background using a sample of events containing two muons, varying the tightness of the requirements on observed tracks in the layers of the silicon detector. The background being searched for should appear as the requirements are loosened. It turns out that such events seem to contain an anomalous component with unexpected properties that *disagree with those of the known possible sources of background. The number of these anomalous events is large (tens of thousands), so this cannot just be a statistical fluctuation.*

One of the anomalous properties of these events is that they contain tracks with large impact parameters, of order a centimeter rather than the hundreds of microns characteristic of b-quark decays. Fitting this tail by an exponential, one gets what one would expect to see from the decay of a new, unknown particle with a lifetime of about 20 picoseconds. These events have further unusual properties, including an anomalously high number of additional muons in small angular cones about the primary ones.

The lifetime is estimated to be considerably longer than b quark life time and below the lifetime 89.5 ps of $K_{0,s}$ mesons. The fit to the tail of "ghost" muons gives the estimate of 20 picoseconds.

3.3.2 The second gift

In October 29 also another remarkable paper [48] had appeared in arXiv. It was titled *Observation of an anomalous positron abundance in the cosmic radiation.* PAMELA collaboration finds an excess of cosmic ray positron at energies $10 \rightarrow 50$ GeV. PAMELA anomaly is discussed in Resonaances blog [56]. ATIC collaboration in turn sees an excess of electrons and positrons going all the way up to energies of order 500-800 GeV [49].

Also Peter Woit refers to these cosmic ray anomalies and also to the article *LHC Signals for a SuperUnified Theory of Dark Matter* by Nima Arkadi-Hamed and Neal Weiner [50], where a model of dark matter inspired by these anomalies is proposed together with a prediction of lepton jets with invariant masses with mass scale of order GeV. The model assumes a new gauge interaction for dark matter particles with Higgs and gauge boson masses around GeV. The prediction is that LHC should detect "lepton jets" with smaller angular separations and GeV scale invariant masses.

3.3.3 Explanation of the CDF anomaly

Consider first the CDF anomaly. TGD predicts a fractal hierarchy of QCD type physics. In particular, colored excitations of leptons are predicted to exist. Neutral lepto-pions would have mass only slightly above two times the charged lepton mass. Also charged lepto-pions are predicts and their masses depend on what is the p-adic mass scale of neutrino and it is not clear whether it is much longer than that for charge colored lepton as in the case of ordinary leptons.

1. There exists a considerable evidence for colored electrons as already found. The anomalous production of electron positron pairs discovered in heavy ion collisions can be understood in terms of decays of electro-pions produced in the strong non-orthogonal electric and magnetic fields created in these collisions. The action determining the production rate would be proportional to the product of the lepto-pion field and highly unique "instanton" action for electromagnetic field determined by anomaly arguments so that the model is highly predictive.

2. Also the .511 MeV emission line [51, 52] from the galactic center can be understood in terms of decays of neutral electro-pions to photon pairs. Electro-pions would reside at magnetic flux tubes of strong galactic magnetic fields. It is also possible that these particles are dark in TGD sense.

3. There is also evidence for colored excitations of muon and muo-pion [53, 54]. Muo-pions could be produced by the same mechanism as electro-pions in high energy collisions of charged particles when strong non-orthogonal magnetic and electric fields are generated.

Also τ-hadrons are possible and CDF anomaly can be understood in terms of a production of higher energy τ-hadrons as the following argument demonstrates.

1. τ-QCD at high energies would produce "lepton jets" just as ordinary QCD. In particular, muon pairs with invariant energy below $2m(\tau) \sim 3.6$ GeV would be produced by the decays of neutral τ-pions. The production of monochromatic gamma ray pairs is predicted to dominate the decays. Note that the space-time sheet associated with both ordinary hadrons and τ lepton correspond to the p-adic prime $M_{107} = 2^{107} - 1$.

2. The model for the production of electro-pions in heavy ion collisions suggests that the production of τ-pions could take place in higher energy collisions of protons generating very strong non-orthogonal magnetic and electric fields. This This would reduce the model to the quantum model for electro-pion production.

3. One can imagine several options for the detailed production mechanism.

   (a) The decay of *virtual* τ-pions created in these fields to pairs of leptobaryons generates lepton jets. Since colored leptons correspond to color octets, lepto-baryons could correspond to states of form LLL or $L\overline{L}L$.

   (b) The option inspired by a blog discussion with Ervin Goldfein is that a coherent state of τ-pions is created first and is then heated to QCD plasma like state producing the lepton jets like in QCD. The linear coupling to $E \cdot B$ defined by em fields of colliding nucleons would be analogous to the coupling of harmonic oscillator to constant force and generate the coherent state.

   (c) The option inspired by CDF model [60] is that a p-adically scaled up variant of *on mass shell* neutral τ-pion having $k = 103$ and 4 times larger mass than $k = 107$ τ-pion is produced and decays to three $k = 105$ τ-pions with $k = 105$ neutral τ-pion in turn decaying to three $k = 107$ τ-pions.

4. The basic characteristics of the anomalous muon pair prediction seems to fit with what one would expect from a jet generating a cascade of τ-pions. Muons with both charges would be produced democratically from neutral τ-pions; the number of muons would be anomalously high; and the invariant masses of muon pairs would be below 3.6 GeV for neutral τ-pions and below 1.8 GeV for charged τ-pions if colored neutrinos are light.

5. The lifetime of 20 ps can be assigned with charged τ-pion decaying weakly only into muon and neutrino. This provides a killer test for the hypothesis. In absence of CKM mixing for colored neutrinos, the decay rate to lepton and its antineutrino is given by

$$\Gamma(\pi_\tau \rightarrow L + \overline{\nu}_L) = \frac{G^2 m(L)^2 f^2(\pi)(m(\pi_\tau)^2 - m(L)^2)^2}{4\pi m^3(\pi_\tau)} . \tag{77}$$

   The parameter $f(\pi_\tau)$ characterizing the coupling of pion to the axial current can be written as $f(\pi_\tau) = r(\pi_\tau)m(\pi_\tau)$. For ordinary pion one has $f(\pi) = 93$ MeV and $r(\pi) = .67$. The decay rate for charged τ-pion is obtained by simple scaling giving

$$\begin{aligned} \Gamma(\pi_\tau \rightarrow L + \overline{\nu}_L) &= 8x^2u^2y^3(1-z^2)\frac{1}{cos^2(\theta_c)}\Gamma(\pi \rightarrow \mu + \overline{\nu_\mu}) , \\ x &= \frac{m(L)}{m(\mu)} , \quad y = \frac{m(\tau)}{m(\pi)} , \quad z = \frac{m(L)}{2m(\tau)} , \quad u = \frac{r(\pi_\tau)}{r(\pi)} . \end{aligned} \tag{78}$$

If the p-adic mass scale of the colored neutrino is same as for ordinary neutrinos, the mass of charged lepto-pion is in good approximation equal to the mass of τ and the decay rates to τ and electron are for the lack of phase space much slower than to muons so that muons are produced preferentially.

6. For $m(\tau) = 1.8$ GeV and $m(\pi) = .14$ GeV and the same value for f_π as for ordinary pion the lifetime is obtained by scaling from the lifetime of charged pion about 2.6×10^{-8} s. The prediction is 3.31×10^{-12} s to be compared with the experimental estimate about 20×10^{-12} s. $r(\pi_\tau) = .41 r_\pi$ gives a correct prediction. Hence the explanation in terms of τ-pions seems to be rather convincing unless one is willing to believe in really nasty miracles.

7. Neutral τ-pion would decay dominantly to monochromatic pairs of gamma rays. The decay rate is dictated by the product of τ-pion field and "instanton" action, essentially the inner product of electric and magnetic fields and reducing to total divergence of instanton current locally. The rate is given by

$$\begin{aligned} \Gamma(\pi_\tau \rightarrow \gamma+\gamma) &= \frac{\alpha_{em}^2 m^3(\pi_\tau)}{64\pi^3 f(\pi_\tau)^2} = 2x^{-2} y \times \Gamma(\pi \rightarrow \gamma+\gamma) \ , \\ x &= \frac{f(\pi_\tau)}{m(\pi_\tau)} \ , \ y = \frac{m(\tau)}{m(\pi)} \ .\Gamma(\pi \rightarrow \gamma+\gamma) = 7.37 \ eV \ . \end{aligned} \tag{79}$$

   The predicted lifetime is 1.17×10^{-17} seconds.

8. Second decay channel is to lepton pairs, with muon pair production dominating for kinematical reasons. The invariant mass of the pairs is 3.6 GeV of no other particles are produced. Whether the mass of colored neutrino is essentially the same as that of charged lepton or corresponds to the same p-adic scale as the mass of the ordinary neutrino remains an open question. If colored neutrino is light, the invariant mass of muon-neutrino pair is below 1.78 GeV.

3.3.4 PAMELA and ATIC anomalies

TGD predicts also a hierarchy of hadron physics assignable to Mersenne primes. The mass scale of M_{89} hadron physics is by a factor 512 higher than that of ordinary hadron physics. Therefore a very rough estimate for the nucleons of this physics is 512 GeV. This suggest that the decays of M_{89} hadrons are responsible for the anomalous positrons and electrons up to energies 500-800 GeV reported by ATIC collaboration. An equally naive scaling for the mass of pion predicts that M_{89} pion has mass 72 GeV. This could relate to the anomalous cosmic ray positrons in the energy interval 10-50 GeV reported by PAMELA collaboration. Be as it may, the prediction is that M_{89} hadron physics exists and could make itself visible in LHC.

The surprising finding is that positron fraction (the ratio of flux of positrons to the sum of electron and positron fluxes) increases above 10 GeV. If positrons emerge from secondary production during the propagation of cosmic ray-nuclei, this ratio should decrease if only standard physics is be involved with the collisions. This is taken as evidence for the production of electron-positron pairs, possibly in the decays of dark matter particles.

Leptohadron hypothesis predicts that in high energy collisions of charged nuclei with charged particles of matter it is possible to produce also charged electro-pions, which decay to electrons or positrons depending on their charge and produce the electronic counterparts of the jets discovered in CDF. This proposal - and more generally leptohadron hypothesis - could be tested by trying to find whether also electronic jets can be found in proton-proton collisions. They should be present at considerably lower energies than muon jets. I decided to check whether I have said something about this earlier and found that I have noticed years ago that there is evidence for the production of anomalous electron-positron pairs in hadronic reactions [21, 22, 23, 24]: some of it dates back to seventies.

The first guess is that the center of mass energy at which the jet formation begins to make itself visible is in a constant ratio to the mass of charged lepton. From CDF data this ratio satisfies $\sqrt{s}/m_\tau = x < 10^3$. For electro-pions the threshold energy would be around $10^{-3}x \times .5$ GeV and for muo-pions around $10^{-3}x \times 100$ GeV.

3.3.5 Comparison of TGD model with the model of CDF collaboration

Few days after the experimental a theoretical paper by CDF collaboration proposing a phenomenological model for the CDF anomaly appeared in the arXiv [60], and it is interesting to compare the model with TGD based model (or rather, one of them corresponding to the third option mentioned above).

The paper proposes that three new particles are involved. The masses for the particles - christened h_3, h_2, and h_1 - are assumed to be 3.6 GeV, 7.3 GeV, and 15 GeV. h_1 is assumed to be pair produced and decay to h_2 pair decaying to h_3 pair decaying to a τ pair.

h_3 is assumed to have mass 3.6 GeV and life-time of 20×10^{-12} seconds. The mass is same as the TGD based prediction for neutral τ-pion mass, whose lifetime however equals to 1.12×10^{-17} seconds ($\gamma + \gamma$ decay dominates). The correct prediction for the lifetime provides a strong support for the identification of long-lived state as charged τ-pion with mass near τ mass so that the decay to μ and its antineutrino dominates. Hence the model is not consistent with leptohadronic model.

p-Adic length scale hypothesis predicts that allowed mass scales come as powers of $\sqrt{2}$ and these masses indeed come in good approximation as powers of 2. Several p-adic scales appear in low energy hadron physics for quarks and this replaces Gell-Mann formula for low-lying hadron masses. Therefore one can ask whether the proposed masses correspond to neutral tau-pion with $p = M_k = 2^k - 1$, $k = 107$, and its p-adically scaled up variants with $p \simeq 2^k$, $k = 105$, and $k = 103$ (also prime). The prediction for masses would be 3.6 GeV, 7.2 GeV, 14.4 GeV.

This co-incidence cannot of course be taken too seriously since the powers of two in CDF model have a rather mundane origin: they follow from the assumed production mechanism producing 8 τ-leptons from h_1. One can however spend some time by looking whether it could be realized somehow allowing p-adically scaled up variants of τ-pion.

1. The proposed model for the production of muon jets is based on production of k=103 neutral τ-pion (or several of them) having 8 times larger mass than k=107 τ-pion in strong EB background of the colliding proton and antiproton and decaying via strong interactions to k=105 and k=107 τ-pions.

2. The first step would be

   $$\pi_\tau^0(103) \rightarrow \pi_\tau^0(105) + \pi_\tau^+(105) + \pi_\tau^-(105) \ .$$

   This step is not kinematically possible if masses are obtained by exact scaling and if $m(\pi_\tau^0) < m(pi_\tau^\pm)$ holds true as for ordinary pion. p-Adic mass formulas do not however predict exact scaling. In the case that reaction is not kinematically possible, it must be replaced with a reaction in which second charged k=105 pion is virtual and decays weakly. This option however reduces the rate of the process dramatically and might be excluded.

3. Second step would consist of a scaled variant of the first step

$$\pi_\tau^0(105) \to \pi_\tau^0(107) + \pi_\tau^+(107) + \pi_\tau^-(107) \ ,$$

where second charged pion also can be virtual and decay weakly, and the weak decays of the $\pi_\tau^\pm(105)$ with mass $2m(\tau)$ to lepton pairs. The rates for these are obtained from previous formulas by scaling.

4. The last step would involve the decays of both charged and neutral $\pi_\tau(107)$. The signature of the mechanism would be anomalous γ pairs with invariant masses $2^k \times m(\tau)$, $k = 1, 2, 3$ coming from the decays of neutral τ-pions.

5. Dimensionless four-pion coupling λ determines the decay rates for neutral τ-pions appearing in the cascade. Rates are proportional to phase space-volumes, which are rather small by kinetic reasons.

The total cross section for producing single lepto-pion can be estimated by using the quantum model for lepto-pion production. Production amplitude is essentially Coulomb scattering amplitude for a given value of the impact parameter b for colliding proton and anti-proton multiplied by the amplitude $U(b,p)$ for producing on mass shell $k = 103$ lepto-pion with given four-momentum in the fields E and B and given essentially by the Fourier transform of $E \cdot B$. The replacement of the motion with free motion should be a good approximation.

UV and IR cutoffs for the impact parameter appear in the model and are identifiable as appropriate p-adic length scales. UV cutoff could correspond to the Compton size of nucleon ($k = 107$) and IR cutoff to the size of the space-time sheets representing topologically quantized electromagnetic fields of colliding nucleons (perhaps $k = 113$ corresponding to nuclear p-adic length scale and size for color magnetic body of constituent quarks or $k = 127$ for the magnetic body of current quarks with mass scale of order MeV). If one has $\hbar/\hbar_0 = 2^7$ one could also guess that the IR cutoff corresponds to the size of dark em space-time sheet equal to $2^7 L(113) = L(127)$ (or $2^7 L(127) = L(141)$), which corresponds to electron's p-adic length scale. These are of course rough guesses.

Quantitatively the jet-likeness of muons means that the additional muons are contained in the cone $\theta < 36.8$ degrees around the initial muon direction. If the decay of $\pi_\tau^0(k)$ can occur to on mass shell $\pi_\tau^0(k+2)$, $k = 103, 105$, it is possible to understand jets as a consequence of the decay kinematics forcing the pions resulting as decay products to be almost at rest.

1. Suppose that the decays to three pions can take place as on mass shell decays so that pions are very nearly at rest. The distribution of decay products $\mu\overline{\nu}$ in the decays of $\pi^\pm(105)$ is spherically symmetric in the rest frame and the energy and momentum of the muon are given by

$$[E,p] = [m(\tau) + \frac{m^2(\mu)}{4m(\tau)}, m(\tau) - \frac{m^2(\mu)}{4m(\tau)}] \ .$$

   The boost factor $\gamma = 1/\sqrt{1-v^2}$ to the rest system of muon is $\gamma = \frac{m(\tau)}{m(\mu)} + \frac{m(\mu)}{4m(\tau)}) \sim 18$.

2. The momentum distribution for μ^+ coming from π_τ^+ is spherically symmetric in the rest system of π^+ . In the rest system of μ^- the momentum distribution is non-vanishing only for when the angle θ between the direction of velocity of μ^- is below a maximum value of given by $tan(\theta_{max}) = 1$ corresponding to a situation in which the momentum μ^+ is orthogonal to the momentum of μ^- (the maximum transverse momentum equals to $m(\mu)v\gamma$ and longitudinal momentum becomes $m(\mu)v\gamma$ in the boost). This angle corresponds to 45 degrees and is not too far from 36.8 degrees.

3. At the next step the energy of muons resulting in the decays of $\pi^{\pm}(103)$

$$[E,p] = [\frac{m(\tau)}{2} + \frac{m^2(\mu)}{2m(\tau)}, \frac{m(\tau)}{2} - \frac{m^2(\mu)}{2m(\tau)}] \ ,$$

and the boost factor is $\gamma_1 = \frac{m(\tau)}{2m(\mu)} + \frac{m(\mu)}{2m(\tau)} \sim 9$. θ_{max} satisfies the condition $tan(\theta_{max}) = \gamma_1 v_1/\gamma v \simeq 1/2$ giving $\theta_{max} \simeq 26.6$ degrees.

If on mass shell decays are not allowed the situation changes since either of the charged pions is off mass shell. In order to obtain similar result the virtual should occur dominantly via states near to on mass shell pion. Since four-pion coupling is just constant, this option does not seem to be realized.

Quantitatively the jet-likeness of muons means that the additional muons are contained in the cone $\theta < 36.8$ degrees around the initial muon direction. If the decay of $\pi^0_\tau(k)$ can occur to on mass shell $\pi^0_\tau(k+2)$, $k = 103, 105$, it is possible to understand jets as a consequence of the decay kinematics forcing the pions resulting as decay products to be almost at rest.

1. Suppose that the decays to three pions can take place as on mass shell decays so that pions are very nearly at rest. The distribution of decay products $\mu\overline{\nu}$ in the decays of $\pi^{\pm}(105)$ is spherically symmetric in the rest frame and the energy and momentum of the muon are given by

$$[E,p] = [m(\tau) + \frac{m^2(\mu)}{4m(\tau)}, m(\tau) - \frac{m^2(\mu)}{4m(\tau)}] \ .$$

The boost factor $\gamma = 1/\sqrt{1-v^2}$ to the rest system of muon is $\gamma = \frac{m(\tau)}{m(\mu)} + \frac{m(\mu)}{4m(\tau} \sim 18$.

2. The momentum distribution for μ^+ coming from π^+_τ is spherically symmetric in the rest system of π^+ . In the rest system of μ^- the momentum distribution is non-vanishing only for when the angle θ between the direction of velocity of μ^- is below a maximum value of given by $tan(\theta_{max}) = 1$ corresponding to a situation in which the momentum μ^+ is orthogonal to the momentum of μ^- (the maximum transverse momentum equals to $m(\mu)v\gamma$ and longitudinal momentum becomes $m(\mu)v\gamma$ in the boost). This angle corresponds to 45 degrees and is not too far from 36.8 degrees.

3. At the next step the energy of muons resulting in the decays of $\pi^{\pm}(103)$

$$[E,p] = [\frac{m(\tau)}{2} + \frac{m^2(\mu)}{2m(\tau)}, \frac{m(\tau)}{2} - \frac{m^2(\mu)}{2m(\tau)}] \ ,$$

and the boost factor is $\gamma_1 = \frac{m(\tau)}{2m(\mu)} + \frac{m(\mu)}{2m(\tau)} \sim 9$. θ_{max} satisfies the condition $tan(\theta_{max}) = \gamma_1 v_1/\gamma v \simeq 1/2$ giving $\theta_{max} \simeq 26.6$ degrees.

If on mass shell decays are not possible, the situation changes since either of the charged pions is off mass shell. In order to obtain similar result the virtual should occur dominantly via states near to on mass shell pion. Since four-pion coupling is just constant, this option does not seem to be realized.

3.3.6 Numerical estimate for the production cross section

The numerical estimate of the cross section involves some delicacies. The model has purely physical cutoffs which must be formulated in a precise manner.

1. Since energy conservation is not coded into the model, some assumption about the maximal τ-pion energy in cm system expressed as a fraction ϵ of proton's center of mass energy is necessary. Maximal fraction corresponds to the condition $m(\pi_\tau) \leq m(\pi_\tau)\gamma_1 \leq \epsilon m_p \gamma_{cm}$ in cm system giving $[m(\pi_\tau)/(m_p\gamma_{cm}) \leq \epsilon \leq 1$. γ_{cm} can be deduced from the center of mass energy of proton as $\gamma_{cm} = \sqrt{s}2m_p$, $\sqrt{s} = 1.96$ TeV. This gives $1.6 \times 10^{-2} < \epsilon < 1$ in a reasonable approximation. It is convenient to parameterize ϵ as

$$\epsilon = (1+\delta) \times \frac{m(\pi_\tau)}{m_p} \times \frac{1}{\gamma_{cm}} \quad .$$

   The coordinate system in which the calculations are carried out is taken to be the rest system of (say) antiproton so that one must perform a Lorentz boost to obtain upper and lower limits for the velocity of τ-pion in this system. In this system the range of γ_1 is fixed by the maximal cm velocity fixed by ϵ and the upper/lower limit of γ_1 corresponds to a direction parallel/opposite to the velocity of proton.

2. By Lorentz invariance the value of the impact parameter cutoff b_{max} should be expressible in terms τ-pion Compton length and the center of mass energy of the colliding proton and the assumption is that $b_{max} = \gamma_{cm} \times \hbar/m(\pi_\tau)$, where it is assumed $m(\pi_\tau) = 8m(\tau)$. The production cross section does not depend much on the precise choice of the impact parameter cutoff b_{max} unless it is un-physically large in which case b^2_{max} proportionality is predicted.

The numerical estimate for the production cross section involves some delicacies.

1. The power series expansion of the integral of CUT_1 using partial fraction representation does not converge since that roots $c_\pm$ are very large in the entire integration region. Instead the approximation $A_1 \simeq iBcos(\psi)/D$ simplifying considerably the calculations can be used. Also the value of b_1L is rather small and one can use stationary phase approximation for CUT_2. It turns out that the contribution of CUT_2 is negligible as compared to that of CUT_1.

2. Since the situation is singular for $\theta = 0$ and $\phi = 0$ and $\phi = \pi/2$ (by symmetry it is enough to calculate the cross section only for this kinematical region), cutoffs

$$\theta \in [\epsilon_1, (1-\epsilon_1)] \times \pi \quad , \quad \phi \in [\epsilon_1, (1-\epsilon_1)] \times \pi/2 \quad , \quad \epsilon_1 = 10^{-3} \quad .$$

   The result of the calculation is not very sensitive to the value of the cutoff.

3. Since the available numerical environment was rather primitive (MATLAB in personal computer), the requirement of a reasonable calculation time restricted the number of intervals in the discretization for the three kinematical variables γ, θ, ϕ to be below $N_{max} = 80$. The result of calculation did not depend appreciably on the number of intervals above $N = 40$ for γ_1 integral and for θ and ϕ integrals even $N = 10$ gave a good estimate.

The calculations were carried for the $exp(iS)$ option since in good approximation the estimate for $exp(iS) - 1$ model is obtained by a simple scaling. $exp(iS)$ model produces a correct order of magnitude for the cross section whereas $exp(iS) - 1$ variant predicts a cross section, which is by several orders of magnitude smaller by downwards α^2_{em} scaling. As I asked Tommaso Dorigo for an estimate for the production cross section in his first blog posting [58], he mentioned that authors refer to a production cross section is 100 nb which looks to me suspiciously large (too large by three orders of magnitude), when compared with the production rate of muon pairs from b-bbar. $\delta = 1.5$ which corresponds to τ-pion energy 36 GeV gives the estimate $\sigma = 351$ nb. The energy is suspiciously high.

In fact, in the recent blog posting of Tommaso Dorigo [57] a value of order .1 nb for the production cross section was mentioned. Electro-pions in heavy ion collisions are produced almost at rest and one has $\Delta v/v \simeq .2$ giving $\delta = \Delta E/m(\pi) \simeq 2 \times 10^{-3}$. If one believes in fractal scaling, this should be at least the order of magnitude also in the case of τ-pion. This would give the estimate $\sigma = 1$ nb. For $\delta = \Delta E/m(\pi) \simeq 10^{-3}$ a cross section $\sigma = .16$ nb would result.

One must of course take the estimate cautiously but there are reasons to hope that large systematic errors are not present anymore. In any case, the model can explain also the order of magnitude of the production cross section under reasonable assumptions about cutoffs.

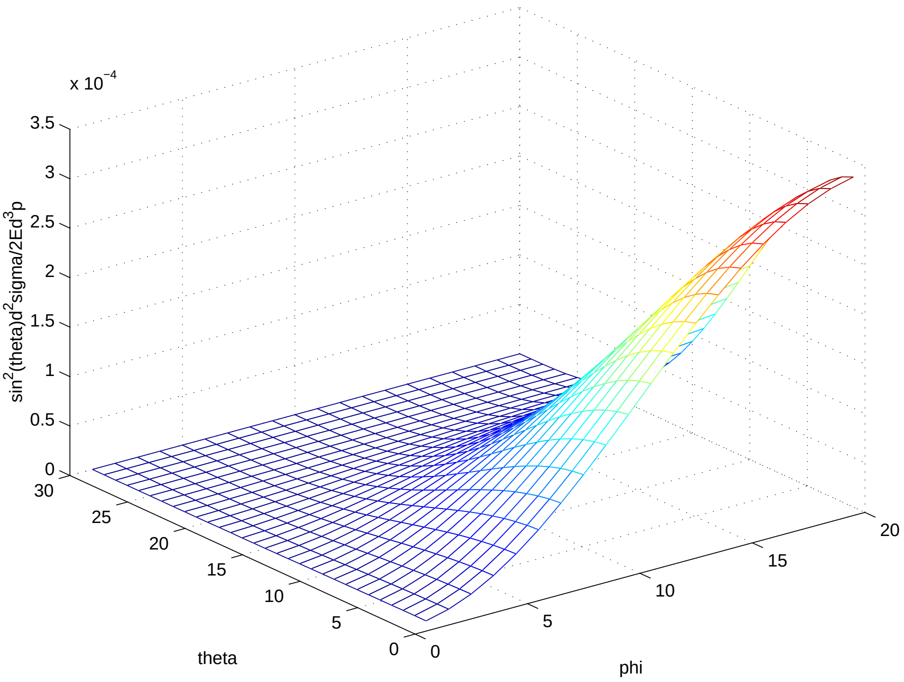

Figure 2: Differential cross section $sin^2(\theta) \times \frac{d^2\sigma}{2Ed^3p}$ for τ-pion production for $\gamma_1 = 1.090 \times 10^3$ in the rest system of antiproton for $\delta = 1.5$. $m(\pi_\tau)$ defines the unit of energy and nb is the unit for cross section. The ranges of θ and ϕ are $(0, \pi)$ and $(0, \pi/2)$.

3.3.7 Does the production of lepto-pions involve a phase transition increasing Planck constant?

The critical argument of Tommaso Dorigo in his blog inspired an attempt to formulate more precisely the hypothesis $\sqrt{s}/m_\tau > x < 10^3$. This led to the realization that a phase transition increasing Planck constant might happen in the production process as also the model for the production of electro-pions requires.

Suppose that the instanton coupling gives rise to *virtual* neutral lepto-pions which ultimately produce the jets (this is first of the three models that one can imagine). E and B could be associated with the colliding proton and antiproton or quarks.

1. The amplitude for lepto-pion production is essentially Fourier transform of $E \cdot B$, where E and B are the non-orthogonal electric and magnetic fields of the colliding charges. At the level of scales one has $\tau \sim \hbar/E$, where τ is the time during which $E \cdot B$ is large enough during collision and E is the energy scale of the virtual lepto-pion giving rise to the jet.

2. In order to have jets one must have $m(\pi_\tau) << E$. If the scaling law $E \propto \sqrt{s}$ hold true, one indeed has $\sqrt{s}/m(\pi_\tau) > x < 10^3$.

3. If proton and antiproton would move freely, τ would be of the order of the time for proton to move through a distance, which is 2 times the Lorentz contracted radius of proton: $\tau_{free} = 2\times\sqrt{1-v^2}R_p/v = 2\hbar/E_p$. This would give for the energy scale of virtual τ-pion the estimate $E = \hbar/\tau_{free} = \sqrt{s}/4$. $x = 4$ is certainly quite too small value. Actually $\tau > \tau_{free}$ holds true but one can argue that without new physics the time for the preservation of $E \cdot B$ cannot be by a factor of order 2^8 longer than for free collision.

4. For a colliding quark pair one would have $\tau_{free} = 4\hbar/\sqrt{s_{pair}(s)}$, where $\sqrt{s_{pair}(s)}$ would be the typical invariant energy of the pair which is exponentially smaller than $\sqrt{s}$. Somewhat paradoxically from classical physics point of view, the time scale would be much longer for the collision of quarks than that for proton and antiproton.

The possible new physics relates to the possibility that lepto-pions are dark matter in the sense that they have Planck constant larger than the standard value.

1. Suppose that the produced lepto-pions have Planck constant larger than its standard value $\hbar_0$. Originally the idea was that larger value of $\hbar$ would scale up the production cross section. It turned out that this is not the case. For $exp(iS)$ option the lowest order contribution is not affected by the scaling of $\hbar$ and for $exp(iS)-1$ option the lowest order contribution scales down as $1/hbar^2$. The improved formulation of the model however led to a correct order of magnitude estimates for the production cross section.

2. Assume that a phase transition increasing Planck constant occurs during the collision. Hence τ is scaled up by a factor $y = \hbar/\hbar_0$. The inverse of the lepto-pion mass scale is a natural candidate for the scaled up dark time scale. $\tau(\hbar_0) \sim \tau_{free}$, one obtains $y \sim \sqrt{s_{min}}/4m(\pi_\tau) \leq 2^8$ giving for proton-antiproton option the first guess $\sqrt{s}/m(\pi_\tau) > x < 2^{10}$. If the value of y does not depend on the type of lepto-pion, the proposed estimates for muo- and electro-pion follow.

3. If the fields E and B are associated with colliding quarks, only colliding quark pairs with $\sqrt{s_{pair}(s)} > (>)m(\pi_\tau)$ contribute giving $y_q(s) = \sqrt{s_{pair}(s)/s} \times y$.

If the τ-pions produced in the magnetic field are on-mass shell τ-pions with $k = 113$, the value of $\hbar$ would satisfy $\hbar/\hbar_0 < 2^5$ and $\sqrt{s}/m(\pi_\tau) > x < 2^7$.

3.3.8 Could it have been otherwise?

To sum up, the probability that a correct prediction for the lifetime of the new particle using only known lepton masses and standard formulas for weak decay rates follows by accident is extremely low. Throwing billion times coin and getting the same result every time might be something comparable to this. Therefore my sincere hope is that colleagues would be finally mature to take TGD seriously. If TGD based explanation of the anomalous production of electron positron pairs in heavy ion collisions would have been taken seriously for fifteen years ago, particle physics might look quite different now.

3.4 Could lepto-hadrons be replaced with bound states of exotic quarks?

Can one then exclude the possibility that electron-hadrons correspond to colored quarks condensed around $k = 127$ hadronic space-time sheet: that is M_{127} hadron physics? There are several objections against this identification.

1. The recent empirical evidence for the colored counterpart of μ and τ supports the view that colored excitations of leptons are in question.
2. The octet character of color representation makes possible the mixing of leptons with lepto-baryons of form $L\nu_L\overline{\nu}_L$ by color magnetic coupling between lepto-gluons and ordinary and colored lepton. This is essential for understanding the production of electron-positron pairs.
3. In the case CDF anomaly also the assumption that colored variant of τ neutrino is very light is essential. In the case of colored quarks this assumption is not natural.

3.5 About the masses of lepto-hadrons

The progress made in understanding of dark matter hierarchy [A9] and non-perturbative aspects of hadron physics [F4, F5] allow to sharpen also the model of lepto-hadrons.

The model for the masses of ordinary hadrons [F4] applies also to the scaled up variants of the hadron physics. The two contributions to the hadron mass correspond to quark contribution and a contribution from super-canonical bosons. For quarks labeled identical p-adic primes mass squared is additive and for quarks labeled by different primes mass is additive. Quark contribution is calculable once the p-adic primes of quarks are fixed.

Super-canonical contribution comes from super-canonical bosons at hadronic space-time sheet labeled by Mersenne prime and is universal if one assumes that the topological mixing of the super-canonical bosons is universal. If this mixing is same as for U type quarks, hadron masses can be reproduced in an excellent approximation if the super-canonical boson content of hadron is assumed to correlate with the net spin of quarks.

In the case of baryons and pion and kaon one must assume the presence of a negative color conformal weight characterizing color binding. The value of this conformal weight is same for all baryons and super-canonical contribution dominates over quark contribution for nucleons. In the case of mesons binding conformal weight can be assumed to vanish for mesons heavier than kaon and one can regard pion and kaon as Golstone bosons in the sense that quark contribution gives the mass of the meson.

This picture generalizes to the case of lepto-hadrons.

1. By the additivity of the mass squared leptonic contribution to lepto-pion mass would be $\sqrt{2}m_e(k)$, where k characterizes the p-adic length scale of colored electron. For $k = 127$ the mass of lepto-pion would would be .702 MeV and too small. For $k = 126$ the mass would be $2m_e = 1.02$ MeV and is very near to the mass of the lepto-pion. Note that for ordinary hadrons quarks can appear in several scaled up variants inside hadrons and the value of k depends on hadron. The prediction for the mass of lepto-ρ would be $m_{\pi_L} + \sqrt{7}m_{127} \simeq 1.62$ MeV ($m_{127} = m_e/\sqrt{5}$).
2. The state consisting of three colored electrons would correspond to leptonic variant of Δ_{++} having charge $q = -3$. The quark contribution to the mass of $\Delta_L \equiv \Delta_{L,3-}$ would be by the additivity of mass squared $\sqrt{3} \times m_e(k = 126) = 1.25$ MeV. If super-canonical particle content is same as for Δ_L, super-canonical contribution would be $m_{SC} = 5 \times m_{127}$, and equal to $m_{SC} = .765$ MeV so that the mass of Δ_L would be $m_{\Delta_L} = 2.34$ MeV. If colored neutrino corresponds to the same p-adic prime as colored electron, also lepto-proton has mass in MeV scale.

4 APPENDIX

4.1 Evaluation of lepto-pion production amplitude

4.1.1 General form of the integral

The amplitude for lepto-pion production with four momentum

$$p = (p_0,\bar{p}) = m\gamma_1(1, vsin(\theta)cos(\phi), vsin(\theta)sin(\phi), vcos(\theta)) \ ,$$
$$\gamma_1 = 1/(1-v^2)^{1/2} \ , \quad (80)$$

is essentially the Fourier component of the instanton density

$$U(b,p) = \int e^{ip\cdot x} E\cdot B d^4x \quad (81)$$

associated with the electromagnetic field of the colliding nuclei.

In order to avoid cumbersome numerical factors, it is convenient to introduce the amplitude $A(b,p)$ as

$$A(b,p) = N_0 \times \frac{4\pi}{Z_1 Z_2 \alpha_{em}} \times U(b,p) \ ,$$
$$N_0 = \frac{(2\pi)^7}{i} \quad (82)$$

Coordinates are chosen so that target nucleus is at rest at the origin of coordinates and colliding nucleus moves along positive z direction in $y = 0$ plane with velocity β. The orbit is approximated with straight line with impact parameter b.

Instanton density is just the scalar product of the static electric field E of the target nucleus and magnetic field B the magnetic field associated with the colliding nucleus, which is obtained by boosting the Coulomb field of static nucleus to velocity β. The flux lines of the magnetic field rotate around the direction of the velocity of the colliding nucleus so that instanton density is indeed non vanishing.

The Fourier transforms of E and B for nuclear charge 4π (chose for convenience) giving rise to Coulomb potential $1/r$ are given by the expressions

$$E_i(k) = N\delta(k_0)k_i/k^2 \ ,$$
$$B_i(k) = N\delta(\gamma(k_0-\beta k_z))k_j\varepsilon_{ijz}e^{ik_xb}/((\frac{k_z}{\gamma})^2+k_T^2) \ ,$$
$$N = \frac{1}{(2\pi)^2} \ . \quad (83)$$

The normalization factor corresponds to momentum space integration measure d^4p. The Fourier transform of the instanton density can be expressed as a convolution of the Fourier transforms of E and B.

$$A(b,p) \equiv = N_0 N_1 \int E(p-k)\cdot B(k) d^4k \ ,$$
$$N_1 = \frac{1}{(2\pi)^4} \ . \quad (84)$$

Where the fields correspond to charges $\pm 4\pi$. In the convolution the presence of two delta functions makes it possible to integrate over k_0 and k_z and the expression for U reduces to a two-fold integral

$$\begin{aligned} A(b,p) &= \beta\gamma \int dk_x dk_y exp(ikxb)(k_x p_y - k_y p_x)/AB \ , \\ A &= (p_z - \frac{p_0}{\beta})^2 + p_T^2 + k_T^2 - 2k_T \cdot p_T \\ B &= k_T^2 + (\frac{p_0}{\beta\gamma})^2 \ , \\ p_T &= (p_x, p_y) \ . \end{aligned} \tag{85}$$

To carry out the remaining integrations one can apply residue calculus.

1. k_y integral is expressed as a sum of two pole contributions
2. k_x integral is expressed as a sum of two pole contributions plus two cut contributions.

4.1.2 k_y-integration

Integration over k_y can be performed by completing the integration contour along real axis to a half circle in upper half plane (see Fig. 4.1.3).

The poles of the integrand come from the two factors A and B in denominator and are given by the expressions

$$\begin{aligned} k_y^1 &= i(k_x^2 + (\frac{p_0}{\beta\gamma})^2)^{1/2} \ , \\ k_y^2 &= p_y + i((p_z - \frac{p_0}{\beta})^2 + p_x^2 + k_x^2 - 2p_x k_x)^{1/2} \ . \end{aligned} \tag{86}$$

One obtains for the amplitude an expression as a sum of two terms

$$A(b,p) = 2\pi i \int e^{ik_x b}(U_1 + U_2) dk_x \ , \tag{87}$$

corresponding to two poles in upper half plane.

The explicit expression for the first term is given by

$$\begin{aligned} U_1 &= RE_1 + iIM_1 \ , \\ RE_1 &= (k_x \frac{p_0}{\beta} y - p_x re_1/2)/(re_1^2 + im_1^2) \ , \\ IM_1 &= (-k_x p_y re_1/2K_1^{1/2} - p_x p_y K_1^{1/2})/(re_1^2 + im_1^2) \ , \\ re_1 &= (p_z - \frac{p_0}{\beta})^2 + p_T^2 - (\frac{p_0}{\beta\gamma})^2 - 2p_x k_x \ , \\ im_1 &= -2K_1^{1/2} p_y \ , \\ K_1 &= k_x^2 + (\frac{p_0}{\beta\gamma})^2 \ . \end{aligned} \tag{88}$$

The expression for the second term is given by

$$
\begin{aligned}
U_2 &= RE_2 + iIM_2 \ , \\
RE_2 &= -((k_x p_y - p_x p_y)p_y + p_x re_2/2)/(re_2^2 + im_2^2) \ , \\
IM_2 &= (-(k_x p_y - p_x p_y) re_2/2K_2^{1/2} + p_x p_y K_2^{1/2})/(re_2^2 + im_2^2) \ , \\
re_2 &= -(p_z - \frac{p_0}{\beta})^2 + (\frac{p_0}{\beta\gamma})^2 + 2p_x k_x + \frac{p_0}{\beta} y - \frac{p_0}{\beta} x \ , \\
im_2 &= 2p_y K_2^{1/2} \ , \\
K_2 &= (p_z - \frac{p_0}{\beta})^2 + \frac{p_0}{\beta} x + k_x^2 - 2p_x k_x \ .
\end{aligned} \tag{89}
$$

A little inspection shows that the real parts cancel each other:$RE_1 + RE_2 = 0$. A further useful result is the identity $im_1^2 + re_1^2 = re_2^2 + im_2^2$ and the identity $re_2 = -re_1 + 2p_y^2$.

4.1.3 k_x-integration

One cannot perform k_x-integration completely using residue calculus. The reason is that the terms IM_1 and IM_2 have cuts in complex plane. One can however reduce the integral to a sum of pole terms plus integrals over the cuts.

The poles of U_1 and U_2 come from the denominators and are in fact common for the two integrands. The explicit expressions for the pole in upper half plane, where integrand converges exponentially are given by

$$
\begin{aligned}
re_i^2 &+ im_i^2 = 0 \ , \ i = 1,2 \ , \\
k_x &= (-b + i(-b^2 + 4ac)^{1/2})/2a \ , \\
a &= 4p_T^2 \ , \\
b &= -4((p_z - \frac{p_0}{\beta})^2 + p_T^2 - (\frac{p_0}{\beta\gamma})^2)p_x \ , \\
c &= ((p_z - \frac{p_0}{\beta})^2 + p_T^2 - (\frac{p_0}{\beta\gamma})^2)^2 + 4(\frac{p_0}{\beta\gamma})^2 p_y^2 \ .
\end{aligned} \tag{90}
$$

A straightforward calculation using the previous identities shows that the contributions of IM_1 and IM_2 at pole have opposite signs and the contribution from poles vanishes identically!

The cuts associated with U_1 and U_2 come from the square root terms K_1 and K_2. The condition for the appearance of the cut is that K_1 (K_2) is real and positive. In case of K_1 this condition gives

$$
k_x = it, \ t \in (0, \frac{p_0}{\beta\gamma}) \ . \tag{91}
$$

In case of K_2 the same condition gives

$$
k_x = p_x + it, \ t \in (0, \frac{p_0}{\beta} - p_z) \ . \tag{92}
$$

Both cuts are in the direction of imaginary axis.

The integral over real axis can be completed to an integral over semi-circle and this integral in turn can be expressed as a sum of two terms (see Fig. 4.1.3).

$$
A(b,p) = 2\pi i (CUT_1 + CUT_2) \ . \tag{93}
$$

The first term corresponds to contour, which avoids the cuts and reduces to a sum of pole contributions. Second term corresponds to the addition of the cut contributions.

In the following we shall give the expressions of various terms in the region $\phi \in [0,\pi/2]$. Using the symmetries

$$\begin{aligned} A(b,p_x,-p_y) &= -A(b,p_x,p_y) \ , \\ A(b,-p_x,-p_y) &= \bar{A}(b,p_x,p_y) \ . \end{aligned} \tag{94}$$

of the amplitude one can calculate the amplitude for other values of ϕ.

The integration variable for cuts is the imaginary part t of complexified k_x. To get a more convenient form for cut integrals one can perform a change of the integration variable

$$\begin{aligned} cos(\psi) &= \frac{t}{(\frac{p_0}{\beta\gamma})} \ , \\ cos(\psi) &= \frac{t}{(\frac{p_0}{\beta} - p_z)} \ , \\ \psi &\in [0,\pi/2] \ . \end{aligned} \tag{95}$$

1. The contribution of the first cut

By a painstaking calculation one verifies that the expression for the contribution of the first cut is given by

$$\begin{aligned} CUT_1 &= D_1 \times \int_0^{\pi/2} exp(-\frac{b}{b_0}cos(\psi))A_1 d\psi \ , \\ D_1 &= -\frac{1}{2}\frac{sin(\phi)}{sin(\theta)} \ , \ b_0 = \frac{\hbar}{m}\frac{\beta\gamma}{\gamma_1} \ , \\ A_1 &= \frac{A + iBcos(\psi)}{cos^2(\psi) + 2iCcos(\psi) + D} \ , \\ A &= sin(\theta)cos(\phi) \ , \ B = K \ , \\ C &= K\frac{cos(\phi)}{sin(\theta)} \ , \ D = -sin^2(\phi) - \frac{K^2}{sin^2(\theta)} \ , \\ K &= \beta\gamma(1 - \frac{v_{cm}}{\beta}cos(\theta)) \ , \ v_{cm} = \frac{2v}{1+v^2} \ . \end{aligned} \tag{96}$$

The definitions of the various kinematical variables are given in previous formulas. The notation is tailored to express that A_1 is rational function of $cos(\psi)$.

1. The exponential $exp(-bcos(\psi)/b_0)$ is very small in the condition

$$cos(\psi) \geq cos(\psi_0) \equiv \frac{\hbar}{mb}\frac{\beta\gamma}{\gamma_1 cos(\phi)} \tag{97}$$

holds true. Here $\hbar = 1$ convention has been given up to make clear that the increase of the Compton length of lepto-pion due to the scaling of $\hbar$ increase the magnitude of the contribution. If the condition $cos(\psi_0) << 1$ holds true, the integral over ψ receives contributions only from narrow range of values near the upper boundary $\psi = \pi/2$ plus the contribution corresponding to the pole of X_1. The practical condition is in terms of critical parameter b_{max} above which exponential approaches zero very rapidly.

2. For $cos(\psi_0) << 1$, that is for $b > b_{max}$ and in the approximation that the function multiplying the exponent is replaced with its value for $\psi = \pi/2$, one obtains for CUT_1 the expression

$$\begin{aligned} CUT_1 &\simeq D_1 A_1(\psi = \pi/2)\frac{\hbar}{mb} \\ &= \frac{1}{2} \times \frac{\beta\gamma}{\gamma_1} \times \frac{\hbar}{mb} \times \frac{sin^2(\theta)cos(\phi)sin(\phi)}{sin^2(\theta)sin^2(\phi) + K^2} \quad . \end{aligned} \tag{98}$$

3. For $cos(\psi_0) >> 1$ exponential factor can be replaced by unity in good approximation and the integral reduces to an integral of rational function of $cos(\psi)$ having the form

$$D_1 \frac{A + iBcos(\psi)}{cos^2(\psi) + 2iC \times cos(\psi) + D} \quad . \tag{99}$$

which can be expressed in terms of the roots $c_\pm$ of the denominator as

$$D_1 \times \sum_\pm \frac{A \mp iBc_\pm}{cos(\psi) - c_\pm} \quad , \quad c_\pm) \quad = \quad -iC \pm \sqrt{-C^2 - D} \quad . \tag{100}$$

Integral reduces to an integral of rational function over the interval $[0, 1]$ by the standard substitution $tan(\psi/2) = t$, $d\psi = 2dt/(1+t^2)$, $cos(\psi) = (1-t^2)/(1+t^2)$, $sin(\psi) = 2t/(1+t^2)$.

$$I \quad = \quad 2D_1 \sum_\pm \int_0^1 dt \frac{A \mp iBc_\pm}{1 - c_\pm - (1 + c_\pm)t^2} \tag{101}$$

This gives

$$I \quad = \quad 2D_1 \sum_\pm \frac{A \mp iBc_\pm}{s_\pm} \times arctan(\frac{1 + c_\pm}{1 - c_\pm}) \quad . \tag{102}$$

$s_\pm$ is defined as $\sqrt{1 - c_\pm^2}$ and one must be careful with the signs. This gives for CUT_1 the approximate expression

$$\begin{aligned} CUT_1 &= D_1 \sum_\pm \frac{sin(\theta)cos(\phi) \mp iKc_\pm}{s_\pm} \times arctan(\frac{1 + c_\pm}{1 - c_\pm}) \quad , \\ c_\pm &= \frac{-iKcos(\phi) \pm sin(\phi)\sqrt{sin^2(\theta) + K^2}}{sin(\theta)} \quad . \end{aligned} \tag{103}$$

Arcus tangent function must be defined in terms of logarithm functions since the argument is complex.

4. In the intermediate region, where the exponential differs from unity one can use expansion in Taylor polynomial to sum over integrals of rational functions of $cos(\psi)$ and one obtains the expression

$$
\begin{aligned}
CUT_1 &= D_1\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\frac{b}{b_0})^n I_n \ , \\
I_n &= \sum_{\pm}(A \mp iBc_{\pm} I_n(c_{\pm}) \ , \\
I_n(c) &= \int_0^{\pi/2}\frac{cos^n(\psi)}{cos(\psi)-c} \ .
\end{aligned}
\tag{104}
$$

$I_n(c)$ can be calculated explicitly by expanding in the integrand $cos(\psi))^n$ to polynomial with respect to $cos(\psi)) - c$, $c \equiv c_{\pm}$

$$
\frac{cos^n(\psi)}{cos(\psi)-c} = \sum_{m=0}^{n-1}\left(\begin{array}{c} n \\ m \end{array}\right) c^m (cos(\psi)-c)^{n-m-1} + \frac{c^n}{cos(\psi)-c} \ .
\tag{105}
$$

After the change of the integration variable the integral reads as

$$
\begin{aligned}
I_n(c) &= \sum_{m=0}^{n-1}\sum_{k=0}^{n-m-1}\left(\begin{array}{c} n \\ m \end{array}\right)\left(\begin{array}{c} n-m-1 \\ k \end{array}\right)(-1)^k(1-c)^{n-m-1-k}(1+c)^k c^m I(k,n-m) \\
&+ \frac{c^n}{1-c}\times log[\frac{\sqrt{1-c}+\sqrt{1+c}}{\sqrt{1-c}-\sqrt{1+c}}] \ , \\
I(k,n) &= 2\int dt\frac{t^{2k}}{(1+t^2)^n} \ .
\end{aligned}
\tag{106}
$$

Partial integration for $I(k,n)$ gives the recursion formula

$$
I(k,n) = -\frac{2^{-n+1}}{n-1} + \frac{2k-1}{2(n-1)} \times I(k-1,n-1) \ .
\tag{107}
$$

The lowest term in the recursion formula corresponds to $I(0, n-k)$, can be calculated by using the expression

$$
\begin{aligned}
(1+t^2)^{-n} &= \sum_{k=0}^{n} c(n,k)[(1+it)^{-k} + (1-it)^{-k}] \ , \\
c(n,k) &= \sum_{l=0}^{n-k-1} c(n-1,k+l)2^{-l-2} + c(n-1,n-1)2^{-n+k-1} \ .
\end{aligned}
\tag{108}
$$

The formula is deducible by assuming the expression to be known for n and multiplying the expression with $(1+t^2)^{-1} = [(1+it)^{-1} + (1-it)^{-1}]/2$ and applying this identity to the resulting products of $(1+it)^{-1}$ and $(1-it)^{-1}$. This gives

$$I(0,n) = -2i \sum_{k=2,n} \frac{c(n,k)}{(k-1)}[1 + 2^{(k-1)/2} sin((k-1)\pi/4)] + c(n,1)log(\frac{1+i}{1-i}) \ . \quad (109)$$

This boils down to the following expression for CUT_1

$$\begin{aligned}
CUT_1 &= D_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\frac{b}{b_0})^n I_n \ , \\
I_n &= \sum_{\pm} [A \mp iBc_{\pm}] I_n(cos(c_{\pm}) \ , \\
I_n(c) &= \sum_{m=1}^{n-1} \sum_{k=0}^{n-m-1} \binom{n}{m} \binom{n-m-1}{k} (1-c)^{n-m-1-k}(1+c)^k c^m I(k, n-m-1) \\
&+ \frac{c^n}{1-c} \times log[\frac{\sqrt{1-c}+\sqrt{1+c}}{\sqrt{1-c}-\sqrt{1+c}}] \ , \\
I(k,n) &= -\frac{2^{-n+1}}{n-1} + \frac{2k-1}{2(n-1)} \times I(k-1, n-1) \ , \\
I(0,n) &= -2i \sum_{k=2}^{n} \frac{c(n,k)}{(k-1)}[1 + 2^{(k-1)/2} sin((k-1)\pi/4)] - c(n,1) \ , \\
c(n,k) &= \sum_{l=0}^{n-k-1} c(n-1, k+l) 2^{-l-2} + c(n-1, n-1) 2^{-n+k-1} \ . \quad (110)
\end{aligned}$$

This expansion in powers of $c_{\pm}$ fails to converge when their values are very large. This happens in the case of τ-pion production amplitude. In this case one typically has however the situation in which the conditions $A_1 \simeq iBcos(\psi)/D$ holds true in excellent approximation and one can write

$$\begin{aligned}
CUT_1 &\simeq i\frac{D_1 B}{D} \times \sum_{n=0,1,...} \frac{(-1)^n}{n! 2^n}(\frac{b}{b_0})^n I_n \times \ , \\
I_n &= \int_0^{\pi/2} cos(\psi)^{n+1} d\psi = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{i^{n-2k}-1}{n+1-2k} \ . \quad (111)
\end{aligned}$$

The denominator X_1 vanishes, when the conditions

$$\begin{aligned}
cos(\theta) &= \frac{\beta}{v_{cm}} \ , \\
sin(\phi) &= cos(\psi) \quad (112)
\end{aligned}$$

hold. In forward direction the conditions express the vanishing of the z-component of the lepto-pion velocity in velocity cm frame as one can realize by noticing that condition reduces to the condition $v = \beta/2$ in non-relativistic limit. This corresponds to the production of lepto-pion with momentum in scattering plane and with direction angle $cos(\theta) = \beta/v_{cm}$.

CUT_1 diverges logarithmically for these values of kinematical variables at the limit $\phi \rightarrow 0$ as is easy to see by studying the behavior of the integral near as K approaches zero so that X_1 approaches zero at $sin(\phi) = cos(\Phi)$ and the integral over a small interval of length $\Delta\Psi$ around $cos(\Psi) = sin(\phi)$ gives a contribution proportional to $log(A + B\Delta\Psi))/B$, $A = K[K - 2isin(\theta)sin^2(\phi)]$ and $B = 2sin(\theta)cos(\phi)[sin(\theta)sin(\phi) - iKcos(\phi)]$. Both A and B vanish at the limit $\phi \rightarrow 0$, $K \rightarrow 0$. The exponential damping reduces the magnitude of the singular contribution for large values of $sin(\phi)$ as is clear form the first formula.

2. The contribution of the second cut

The expression for CUT_2 reads as

$$
\begin{aligned}
CUT_2 &= D_2 exp(-\frac{b}{b_2}) \times \int_0^{\pi/2} exp(i\frac{b}{b_1}cos(\psi))A_2 d\psi \ , \\
D_2 &= -\frac{sin(\frac{\phi}{2})}{usin(\theta)} \ , \\
b_1 &= \frac{\hbar}{m}\frac{\beta}{\gamma_1} \ , \ b_2 = \frac{\hbar}{mb}\frac{1}{\gamma_1 \times sin(\theta)cos(\phi)} \\
A_2 &= \frac{Acos(\psi) + B}{cos^2(\psi) + 2iCcos(\psi) + D} \ , \\
A &= sin(\theta)cos(\phi)u \ , \ B = \frac{w}{v_{cm}} + \frac{v}{\beta}sin^2(\theta)[sin^2(\phi) - cos^2(\phi)] \ , \\
C &= \frac{\beta w}{uv_{cm}}\frac{cos(\phi)}{sin(\theta)} \ , \ D = -\frac{1}{u^2}(\frac{sin^2(\phi)}{\gamma^2} + \beta^2(v^2 sin^2(\theta) - \frac{2vw}{v_{cm}})cos^2(\phi)) \\
&+ \frac{w^2}{v_{cm}^2 u^2 sin^2(\theta)} + 2i\frac{\beta v}{u}sin(\theta)cos(\phi) \ , \\
u &= 1 - \beta vcos(\theta) \ , \ w = 1 - \frac{v_{cm}}{\beta}cos(\theta) \ . \qquad (113)
\end{aligned}
$$

$$(114)$$

The denominator X_2 has no poles and the contribution of the second cut is therefore always finite.

1. The factor $exp(-b/b_2)$ gives an exponential reduction and the contribution of CUT_2 is large only when the criterion

$$b < \frac{\hbar}{m} \times \frac{1}{v\gamma_1 sin(\theta)cos(\phi)}$$

   for the impact parameter b is satisfied. Large values of $\hbar$ increase the range of allowed impact parameters since the Compton length of lepto-pion increases.

2. At the limit when the exponent becomes very large the variation of the phase factor implies destructive interference and one can perform stationary phase approximation around $\psi = \pi/2$. This gives

$$
\begin{aligned}
CUT_2 &\simeq \sqrt{\frac{2\pi b_1}{b}} \times D_2 \times exp(\frac{b}{b_2})A_2(\psi = 0) \ , \\
D_2 &= -\frac{sin(\frac{\phi}{2})}{usin(\theta)} \ , \ A_2 = \frac{A}{D} \ . \qquad (115)
\end{aligned}
$$

3. As for CUT_1, the integral over ψ can be expressed as a finite sum of integrals of rational functions, when the value of $(b/b_1)cos(\psi)$ is so small that $exp(i(b/b_1)cos(\psi))$ can be approximated by a Taylor polynomial. More generally, one obtains the expansion

$$\begin{aligned} CUT_2 &= D_2 exp(-\frac{b}{b_2}) \times \sum_{n=0}^{\infty} \frac{1}{n!} i^n (\frac{b}{b_1})^n I_n(A,B,C,D) \ , \\ I_n(A,B,C,D) &= \int_0^{\pi/2} cos(\psi)^n \frac{A + iBcos(\psi)}{cos^2(\psi) + Ccos(\psi) + D} \ . \end{aligned} \quad (116)$$

The integrand of $I_n(A,B,C,D)$ is same rational function as in the case of CUT_1 but the parameters A,B,C,D given in the expression for CUT_2 are different functions of the kinematical variables. The functions appearing in the expression for integrals $I_n(c)$ correspond to the roots of the denominator of A_2 and are given by $c_{\pm} = -iC \pm \sqrt{-C^2 - D}$, where C and D are the function appearing in the general expression for CUT_2 in Eq. 114.

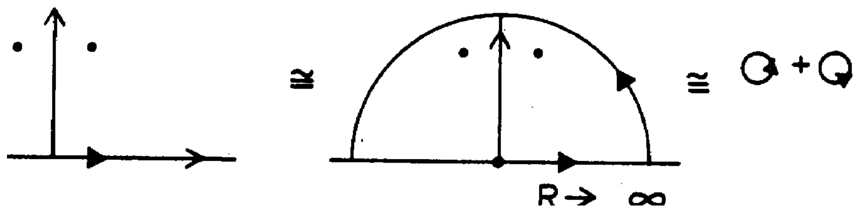

Figure 3: Evaluation of k_y-integral using residue calculus.

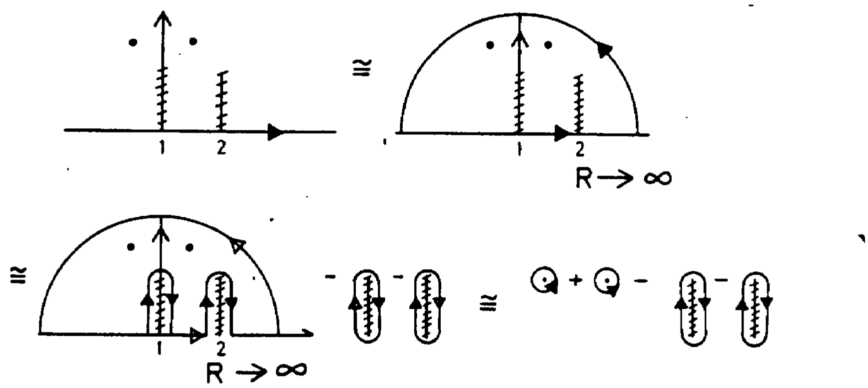

Figure 4: Evaluation of k_x-integral using residue calculus.

4.2 Production amplitude in quantum model

The previous expressions for CUT_1 and CUT_2 as such give the production amplitude for given b in the classical model and the cross section can be calculated by integrating over the values of b. The finite Taylor expansion of the amplitude in powers of b allows explicit formulas when impact parameter cutoff is assumed.

4.2.1 General expression of the production amplitude

In quantum model the production amplitude can be reduced to simpler form by using the defining integral representation of Bessel functions

$$\begin{aligned}
f_B &= i\int F(b)J_0(\Delta kb)(CUT_1+CUT_2)bdb \ , \\
F &= 1 \quad \text{for } exp(i(S)) \text{ option} \ , \\
F(b\geq b_{cr}) &= \int dz\frac{1}{\sqrt{z^2+b^2}} = 2ln(\frac{\sqrt{a^2-b^2}+a}{b}) \ \text{ for } exp(i(S))-1 \text{ option} \ , \\
\Delta k &= 2ksin(\frac{\alpha}{2}) \ , \quad k=M_R\beta \quad . \qquad (117)
\end{aligned}$$

Note that F is a rather slowly varying function of b and in good approximation can be replaced by its average value $A(b,p)$, which has been already explicitly calculated as power series in b. α_{em} corresponds to the value of α_{em} for the standard value of Planck constant.

4.2.2 The limit $\Delta k=0$

The integral of the contribution of CUT_1 over the impact parameter b involves integrals of the form

$$\begin{aligned}
J_{1,n} &= b_0^2\int J_0(\Delta kb)F(b)x^{n+1}dx \ , \\
x &= \frac{b}{b_0} \ . \qquad (118)
\end{aligned}$$

Here a is the upper impact parameter cutoff. For CUT_2 one has integrals of the form

$$\begin{aligned}
J_{2,n} &= b_1^2(\frac{b_2}{b_1})^{n+2}\int J_0(\Delta kb)F(b)exp(-x)x^{n+1}dx \ , \\
x &= \frac{b}{b_2} \ . \qquad (119)
\end{aligned}$$

Using the following approximations it is possible to estimate the integrals analytically.

1. The logarithmic term is slowly varying function and can be replaced with its average value

$$F(b) \rightarrow \langle F(b)\rangle \equiv F \ . \qquad (120)$$

2. Δk is fixed once the value of the impact parameter is known. At the limit $\Delta k=0$ making sense for very high energy collisions one can but the value of Bessel function to $J_0(0)=1$. Hence it is advantageous to calculated the integrals of $\int CUT_ibdb$.

Consider first the integral $\int CUT_1 bdb$. If exponential series converges rapidly one can use Taylor polynomial and calculate the integrals explicitly. When this is not the case one can calculate integral approximately and the total integral is sum over two contributions:

$$\int CUT_1 bdb = I_a + I_b \ . \tag{121}$$

1. The region in which Taylor expansion converges rapidly gives rise integrals

$$\begin{aligned} I_{1,n} &\simeq b_0^2 \int x^{n+1} dx = b_0^2 \frac{1}{n+2}[(\frac{b_{max}}{b_0})^{n+2} - (\frac{b_{cr}}{b_0})^{n+2}] \simeq b_0^2 \frac{1}{n+2}(\frac{b_{max}}{b_0})^{n+2} \ , \\ I_{2,n} &\simeq b_1^2 (\frac{b_2}{b_1})^{n+2} \int exp(-x) x^{n+1} dx = b_1^2 (\frac{b_2}{b_1})^{n+2} (n+1)! \ . \end{aligned} \tag{122}$$

2. For the perturbative part of CUT_1 one obtains the expression

$$\begin{aligned} I_a &= \int_0^{b_{max}} CUT_1 bdb = D_1 \times b_0^2 \times \sum_{n=0}^{\infty} \frac{1}{n!(n+2)} (\frac{b_{max}}{b_0})^{n+2} I_n(A,B,C,D) \ , \\ D_1 &= -\frac{1}{2} \frac{sin(\phi)}{sin(\theta)} \ , \ b_0 = \frac{\hbar \beta \gamma}{m \gamma_1} \ . \end{aligned} \tag{123}$$

   There b_{max} is the largest value of b for which the series converges sufficiently rapidly.

3. The convergence of the exponential series is poor for large values of b/b_0, that is for $b > b_m$. In this case one can use the approximation in which the multiplier of exponent function in the integrand is replaced with its value at $\psi = \pi/2$ so that amplitude becomes proportional to b_0/b. In this case the integral over b gives a factor proportional to ab_0, where a is the impact parameter cutoff.

$$\begin{aligned} I_b \equiv \int_{b_m}^{a} CUT_1 bdb &\simeq b_0(a - b_m) D_1 \times A_1(\psi = \pi/2) \\ &= \frac{\beta\gamma}{\gamma_1} \times \frac{\hbar}{m} \times \frac{sin^2(\theta) cos(\phi) sin(\phi)}{sin^2(\theta) sin^2(\phi) + K^2} \ , \\ D_1 &= -\frac{1}{2} \frac{sin(\phi)}{sin(\theta)} \ , \ A_1(\psi = \pi/2) = \frac{A}{D} \ . \end{aligned} \tag{124}$$

4. As already explained, the expansion based on partial fractions does not converge, when the roots $c_\pm$ have very large values. This indeed occurs in the case of τ-pion production cross section. In this case one has $A_1 \simeq iBcos(\psi)/D$ in excellent approximation and one can calculate CUT_1 in much easier manner. Using the formula of Eq. 111 for CUT_1, one obtains

$$
\begin{aligned}
\int CUT_1 bdb &\simeq b_0^2 \frac{D_1 B}{D} \times \sum_{n=0,1,...} \frac{(-1)^n}{n!(n+2)2^n} \times \sum_{k=0}^{n+1} \left(\begin{array}{c} n+1 \\ k \end{array} \right) c_{n,k} \times (\frac{b_{max}}{b_0})^n \ , \\
c_{n,k} &= \frac{i^{n+1-2k}-1}{n+1-2k} for \ \ n \neq 2k-1 \ \ , \ \ c_{n,k} = \frac{i\pi}{2} \ \ for \ \ n = 2k-1 \ \ , \qquad (125)
\end{aligned}
$$

Note that for $n = 2k + 1 = k$ the coefficient diverges formally and actua

Highly analogous treatment applies to the integral of CUT_2.

1. For the perturbative contribution to $\int CUT_2 bdb$ one obtains

$$
\begin{aligned}
I_a &= \int_0^{b_{1,max}} CUT_2 bdb = b_1^2 D_2 \sum_{n=0}^{\infty} (n+1) i^n I_n(A,B,C,D) \times (\frac{b_2}{b_1})^{n+2} \ , \\
D_2 &= -\frac{sin(\frac{\phi}{2})}{usin(\theta)} \ , \\
b_1 &= \frac{\hbar\beta}{m\gamma_1} \ \ , \ \ b_2 = \frac{\hbar}{m\gamma_1} \frac{1}{sin(\theta)cos(\phi)} \ . \qquad (126)
\end{aligned}
$$

2. Taylor series converges slowly for

$$
\frac{b_1}{b_2} = \frac{sin(\theta)cos(\phi)}{\beta} \rightarrow 0 \ .
$$

   In this case one can replace $exp(-b/b_2)$ with unity or expand it as Taylor series taking only few terms. This gives the expression for the integral which is of the same general form as in the case of CUT_1

$$
I_a = \int_0^{b_{max}} CUT_2 bdb = b_1^2 D_2 \sum_{n=0}^{\infty} \frac{i^n}{n!(n+2)} I_n(A,B,C,D)(\frac{b_{max}}{b_1})^{n+1} \ . \qquad (127)
$$

3. Also when b/b_1 becomes very large, one must apply stationary phase approximation to calculate the contribution of CUT_2 which gives a result proportional to $\sqrt{b_1/b}$. Assume that $b_m >> b_1$ is the value of impact parameter above which stationary phase approximation is good. This gives for the non-perturbative contribution to the production amplitude the expression

$$
\begin{aligned}
I_b &= \int_{b_m}^{a} CUT_2 bdb = k\sqrt{\frac{2\pi b_1}{b_2}} b_2^2 \times D_2 \times A_2(\psi = \pi/2) \ , \\
k &= \int_{x_1}^{x_2} exp(-x) x^{1/2} dx = 2\int_{\sqrt{x_1}}^{\sqrt{x_2}} exp(-u^2) u^2 du \ , \\
x_1 &= \frac{b_m}{b_2} \ \ , \ \ x_2 = \frac{a}{b_2} \ . \qquad (128)
\end{aligned}
$$

   In good approximation one can take $x_2 = \infty$. $x_1 = 0$ gives the upper bound $k \leq \sqrt{\pi}$ for the integral.

Some remarks relating to the numerics are in order.

1. The contributions of both CUT_1 and CUT_2 are proportional to $1/sin(\theta)$ in the forward direction. The denominators of A_i however behave like $1/sin^2(\theta)$ at this limit so that the amplitude behaves as $sin(\theta)$ at this limit and the amplitude approaches to zero like $sin(\theta)$ Therefore the singularity is only apparent but must be taken into account in the calculation since one has $c_\pm \rightarrow i\infty$ at this limit for CUT_2 and for CUT_1 the roots approach to $c_+ = c_- = i\infty$. One must pose a cutoff θ_{min} below which the contribution of CUT_1 and CUT_2 are calculated directly using approximate he expressions for D_iA_i.

$$\begin{aligned} D_1A_1 &\rightarrow -\frac{i}{K}cos(\psi) \times sin(\theta) \rightarrow 0 \\ D_2A_2 &\rightarrow -\frac{uv_{cm}}{w} \times sin(\theta) \rightarrow 0 \ . \end{aligned} \tag{129}$$

   In good approximation both contributions vanish since also $sin^2(\theta)$ factor from the phase space integration reduces the contribution.

2. A second numerical problem is posed by the possible vanishing of

$$K = \beta\gamma(1 - \frac{v_c m}{\beta}cos(\theta)) \ .$$

   In this case the roots $c_\pm = \pm sin(\phi)$ are real and c_+ gives rise to a pole in the integrand.

   The singularity to the amplitude comes from the logarithmic contributions in the Taylor series expansion of the amplitude. The sum of the singular contributions coming from c_+ and c_- are of form

$$\frac{c_n}{2}(\sqrt{1 - sin(\phi)} + \sqrt{1 + sin(\phi)}log(\frac{1+u}{1-u}) \ , \ u = \sqrt{\frac{1 + sin(\phi)}{1 - sin(\phi)}} \ .$$

   Here c_n characterizes the $1/(cos(\psi) - c_\pm)$ term of associated with the $cos(\psi)^n$ term in the Taylor expansion. Logarithm becomes singular for the two terms in the sum at the limit $\phi \rightarrow 0$. The sum however behaves as

$$\frac{c_n}{2}sin(\phi)log(\frac{sin(\phi)}{2}) \ .$$

   so that the net result vanishes at the limit $\phi \rightarrow 0$. It is essential that the logarithmic singularities corresponding to the roots c_+ and c_- cancel each other and this must be taken into account in numerics. There is also apparent singularity at $\phi = \pi/2$ canceled by $cos(\phi)$ factor in D_1. The simplest manner to get rid of the problem is to exclude small intervals $[0, \epsilon]$ and $[\pi/2 - \epsilon, \pi/2]$ from the phase space volume.

4.2.3 Improved approximation to the production cross section

The approximation $J_0(\Delta k_T(b)b) = 1$ and $F(b) = F = constant$ allows to perform the integrations over impact parameter explicitly (for $exp(iS)$ option $F = 1$ holds true identically in the lowest order approximation). An improved approximation is obtained by diving the range of impact parameters to pieces and performing the integrals over the impact parameter ranges exactly using the average values of these functions. This requires only a straightforward generalization of the formulas derived above involving integrals of the functions x^n and $exp(-x)x^n$ over finite range. Obviously this is still numerically well-controlled procedure.

4.3 Evaluation of the singular parts of the amplitudes

The singular parts of the amplitudes $CUT_{1,sing}$ and $B_{1,sing}$ are rational functions of $cos(\psi)$ and the integrals over ψ can be evaluated exactly.

In the classical model the expression for $U_{1,sing}$ appearing as integrand in the expression of $CUT_{1,sing}$ reads as

$$
\begin{aligned}
A_{1,sing} &= -\frac{1}{2\sqrt{K^2+sin^2(\theta)}}(sin(\theta)cos(\phi)A_a + iKA_b) \ , \\
A_a &= I_1(\beta,\pi/2) = \int_0^{\pi/2} d\psi f_1 \ , \\
A_b &= I_2(\beta,\pi/2) = \int_0^{\pi/2} d\psi f_2 \ , \\
f_1 &= \frac{1}{(cos(\psi)-c_1)(cos(\psi)-c_2)} \ , \\
f_2 &= cos(\psi) f_1 \ , \\
c_1 &= \frac{-iKcos(\phi) + sin(\phi)\sqrt{K^2+sin^2(\theta)}}{sin(\theta)} \ , \\
c_2 &= -\bar{c}_1 \ .
\end{aligned} \tag{130}
$$

Here c_i are the roots of the polynomial X_1 appearing in the denominator of the integrand.

In quantum model the approximate expression for the singular contribution to the production amplitude can be written as

$$
\begin{aligned}
B_{1,sing} &\simeq k_1 \frac{sin(\theta)sin(\phi)}{2\sqrt{K^2+sin^2(\theta)}} \sum_n \langle F\rangle_n (I(x(n+1)) - I(x(n))) \ , \\
I(x) &= exp(-\frac{sin(\phi)x}{sin(\phi_0)})(sin(\theta)cos(\phi)A_a(\Delta ka,x) + iKA_b(\Delta ka,x)) \ , \\
k_1 &= 2\pi^2 M_R Z_1 Z_2 \alpha_{em} \frac{\sqrt{2}}{\sqrt{\Delta k\pi}} sin(\phi_0) \ .
\end{aligned} \tag{131}
$$

The expressions for the amplitudes $A_a(k,x)$ and $A_b(k,x)$ read as

$$
\begin{aligned}
A_a(k,x) &= cos(kx)I_3(k,0,\pi/2) + isin(\phi_0)ksin(kx)I_5(k,0,\pi/2) \ , \\
A_b(k,x) &= cos(kx)I_4(k,0,\pi/2) + isin(\phi_0)ksin(kx)I_3(k,0,\pi/2) \ , \\
I_i(k,\alpha,\beta) &= \int_\alpha^\beta f_i(k) d\psi \ , \\
f_3(k) &= \frac{cos(\psi)}{(cos^2(\psi)+sin^2(\phi_0)k^2)} f_1(k) \ , \\
f_4(k) &= cos(\psi) f_3(k) \ , \\
f_5(k) &= \frac{1}{(cos^2(\psi)+sin^2(\phi_0)k^2)} f_1(k) \ .
\end{aligned} \tag{132}
$$

The expressions for the integrals I_i as functions of the endpoints α and β can be written as

$$\begin{aligned}
I_1(k,\alpha,\beta) &= I_0(c_1,\alpha,\beta) - I_0(c_2,\alpha,\beta) \ , \\
I_2(\alpha,\beta) &= c_1 I_0(c_1,\alpha,\beta) - c_2 I_0(c_2,\alpha,\beta) \ , \\
I_3 &= C_{34} \sum_{i=1,2,j=3,4} \frac{1}{(c_i - c_j)} (c_i I_0(c_i,\alpha,\beta) - c_j I_0(c_j,\alpha,\beta)) \ , \\
I_4 &= C_{34} \sum_{i=1,2,j=3,4} \frac{1}{(c_i - c_j)} ((c_i - c_j)(\beta-\alpha) - c_i^2 I_0(c_i,\alpha,\beta) + c_j^2 I_0(c_j,\alpha,\beta)) \ , \\
I_5 &= C_{34} \sum_{i=1,2,j=3,4} \frac{1}{(c_i - c_j)} (I_0(c_i,\alpha,\beta) - I_0(c_j,\alpha,\beta)) \ , \\
C_{34} &= \frac{1}{c_3 - c_4} = \frac{1}{2ikasin(\phi_0)} \ .
\end{aligned} \tag{133}$$

The parameters c_1 and c_2 are the zeros of X_1 as function of $cos(\psi)$ and c_3 and c_4 the zeros of the function $cos^2(\psi) + k^2a^2sin^2(\phi_0)$:

$$\begin{aligned}
c_1 &= \frac{-iKcos(\phi) + sin(\phi)\sqrt{K^2 + sin^2(\theta)}}{sin(\theta)} \ , \\
c_2 &= \frac{-iKcos(\phi) - sin(\phi)\sqrt{K^2 + sin^2(\theta)}}{sin(\theta)} \ , \\
c_3 &= ikasin(\phi_0) \ , \\
c_4 &= -ikasin(\phi_0) \ .
\end{aligned} \tag{134}$$

The basic integral $I_0(c,\alpha,\beta)$ appearing in the formulas is given by

$$\begin{aligned}
I_0(c,\alpha,\beta) &= \int_\alpha^\beta d\psi \frac{1}{(cos(\psi) - c)} \ , \\
&= \frac{1}{\sqrt{1-c^2}} (f(\alpha) - f(\beta)) \ , \\
f(x) &= ln(\frac{(1 + tan(x/2)t_0)}{(1 - tan(x/2)t_0)}) \ , \\
t_0 &= \sqrt{\frac{1-c}{1+c}} \ .
\end{aligned} \tag{135}$$

From the expression of I_0 one discovers that scattering amplitude has logarithmic singularity, when the condition $tan(\alpha/2) = 1/t_0$ or $tan(\beta/2) = 1/t_0$ is satisfied and appears, when c_1 and c_2 are real. This happens at the cone $K = 0$ ($\theta = \theta_0$), when the condition

$$\begin{aligned}
\sqrt{\frac{(1 - sin(\phi))}{(1 + sin(\phi))}} &= tan(x/2) \ , \\
x &= \alpha \text{ or } \beta \ .
\end{aligned} \tag{136}$$

holds true. The condition is satisfied for $\phi \simeq x/2$. $x = 0$ is the only interesting case and gives singularity at $\phi = 0$. In the classical case this gives logarithmic singularity in production amplitude for all scattering angles.

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